Supersymmetry in Enumerative Geometry

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1 Introduction

Supersymmetry concerns the distinction between bosonic and fermionic quantum fields. Essentially, bosonic fields (or bosons) commute with all fields, while fermionic fields (or fermions) anti-commute with each other, meaning $\alpha\beta = -\beta\alpha$. It turns out that anti-commuting behavior has many mathematical benefits. For example, since a fermion anti-commutes with itself, it must square to 0. It follows that power series expansions in a fermionic variable will terminate after two terms. Another example, which is outside the scope of this paper, is that the identity component of the matrix supergroup $GL(1|1)$ admits a parametrization given by a decomposition into lower triangular * diagonal * upper triangular.[1] More familiar matrix groups, such as $GL(2)$ or $SL(2)$, do not have this property.

The basic setup for supersymmetric quantum field theory is as follows. Spacetime is given as a manifold X with n "space" dimensions and one "time" dimension. The corresponding theory is $(n+1)$-dimensional; often times the value of n is even written explicitly, i.e. it may be written "$(2+1)$-dimensional" if $n = 2$. A field on X is something which can be locally expressed using local coordinates on X. The two main examples are sections of vector bundles on X and maps from X to another manifold. Given a set of fields $\mathcal{F}$, the physical theory is given by an action functional $S : \mathcal{F} \to \mathbb{R}$. Classical field theory is done by minimizing S. In quantum field theory, we have path integrals

$$\int \mathcal{D}f e^{iS(f)}$$

where $\mathcal{D}f$ indicates that the integration is over the space of fields $f \in \mathcal{F}$. This is technically not well-defined, but physicists have ways of extracting the information they need from this starting point.

In this paper, we will only be dealing with 0-dimensional quantum field theory. In particular, will explain how the formalism of supersymmetric 0-dimensional quantum field theory can be used to count critical points of functions.

[1] For more information about this example, see [2].

2 0-dimensional Quantum Field Theory

In 0-dimensional quantum field theory (QFT), there is no time variable. We will take the spacetime to be a single point. A field X is then determined by its value $X(p) = x$ at that point. We treat these fields as variables. Notice that constants are also bosons. The action S is just a function of x, and path integrals are ordinary (Riemann) integrals.

We define the partition function

$$Z = \int e^{-S(x)} dx.$$

Notice that we have replaced $iS(x)$ by $-S(x)$ in the argument of the exponential function. This is a procedure known as Wick rotation, which is done in order to make the path integral "more convergent". The original path integral can be obtained by analytic continuation, but we will not elaborate more on this.

3 Supersymmetry in 0-dimensions

In 0-dimensional spacetime, fermions are given by variables which anti-commute with each other, while commuting with constants and bosonic fields. In particular, a fermion ψ satisfies $\psi\psi = -\psi\psi$, so $\psi^2 = 0$. As a consequence, if f is a function depending on a fermion ψ which admits a power series expansion near 0, we have $f(\psi) = f(0) + f'(0)\psi$. We will henceforth assume that all functions of a fermionic variable are of the form $a + b\psi$.

Also, notice that a product of two fermions is a boson: $\psi_1\psi_2\psi_3 = -\psi_1\psi_3\psi_1 = \psi_3\psi_1\psi_2$. These bosons also square to 0, since $\psi_1\psi_2\psi_1\psi_2 = -\psi_1^2\psi_2^2 = 0$.

In supersymmetric QFT, it is required that the action S is bosonic. It is equivalent to require that every term in S contains an even number of fermions. Therefore, we will take the smallest example and work with two fermions, ψ_1 and ψ_2. We will also only work with one boson x.

Fermionic integration is given by the rule

$$\int (a + b\psi) d\psi = b.$$

The precise formalism behind rule is called Berezin integration. There is a general rule for changing variables in an integral, but we only give a special case: if $\tilde{\psi} = a\psi$ for some non-zero boson a, then $d\tilde{\psi} = a^{-1} d\psi$.

We will also need to take integrals with respect to multiple fermionic variables. An arbitrary function in ψ_1, ψ_2 is given by $f(\psi_1, \psi_2) = f_0 + f_1\psi_1 + f_2\psi_2 + f_3\psi_1\psi_2$, and we define

$$\int f(\psi_1, \psi_2) d\psi_1 d\psi_2 = f_3.$$

There is also a corresponding change of variables formula. Suppose $\psi_1' = a\psi_1 + b\psi_2, \psi_2' = c\psi_1 + d\psi_2$. We have

$$f_3 \psi_1' \psi_2' = f_3(a\psi_1 + b\psi_2)(c\psi_1 + d\psi_2) = f_3(ad - bc)\psi_1\psi_2,$$

so

$$\int f(a\psi_1 + b\psi_2, c\psi_1 + d\psi_2) d\psi_1 d\psi_2 = (ad - bc) \int f(\psi_1', \psi_2') d\psi_1' \psi_2'.$$

Given smooth functions f, g, h in x, we define

$$f(g(x) + h(x)\psi_1\psi_2) = f(g(x)) + f'(g(x))h(x)\psi_1\psi_2.$$

For instance, if $f(x) = \exp(x)$, we get

$$\begin{aligned} \exp(g(x) + h(x)\psi_1\psi_2) &= \exp(g(x)) + \exp(g(x))h(x)\psi_1\psi_2 \\ &= \exp(g(x))(1 + h(x)\psi_1\psi_2). \end{aligned}$$

4 A Supersymmetric Action

Let $h(x)$ be a twice differentiable function of x, and let

$$S(x, \psi_1, \psi_2) = \frac{h'(x)^2}{2} - h''(x)\psi_1\psi_2.$$

The partition function is

$$\begin{aligned} Z &= \int \exp(-S(x, \psi_1, \psi_2)) dx d\psi_1 d\psi_2 \\ &= \int \exp(-h'(x)^2/2 + h''(x)\psi_2\psi_2) dx d\psi_1 d\psi_2 \\ &= \int \exp(-h'(x)^2/2)(1 + h''(x)\psi_1\psi_2) dx d\psi_1 d\psi_2 \\ &= \int \exp(-h'(x)^2/2) h''(x) dx. \end{aligned}$$

Suppose h is a nonconstant polynomial function. Using a change of variables $y = h'(x)$, we get

$$Z = \int_{h'(-\infty)}^{h'(\infty)} e^{-y^2/2} dy = c\sqrt{\pi},$$

where c is 0 if h is odd degree, and otherwise c is the sign of the leading coefficient of h. The $\sqrt{\pi}$ can be incorporated into the ambiguous $\mathcal{D}f$ term in the path integral definition, so that the partition function is an integer.

Let us now discuss supersymmetries. Supersymmetries are certain infinitesimal symmetries of the action. We continue working with the same action S as before. A corresponding supersymmetry transformation is given by the rules

$$\begin{aligned}\delta x &= \epsilon_1\psi_1 + \epsilon_2\psi_2,\\ \delta\psi_1 &= h'(x)\epsilon_2,\\ \delta\psi_2 &= -h'(x)\epsilon_1,\end{aligned}$$

where the ϵ_i are two fermionic parameters. We compute the change in S under this transformation, using the additional rules $\delta(f(x)) = f'(x)\delta x$ and $\delta(ab) = (\delta a)b + a(\delta b)$:

$$\begin{aligned}\delta S &= \delta(h'(x)^2/2) - \delta(h''(x)\psi_1\psi_2)\\ &= h'(x)h''(x)\delta x - h'''(x)(\delta x)\psi_1\psi_2 - h''(x)(\delta\psi_1)\psi_2 - h''(x)\psi_1(\delta\psi_2)\\ &= (h'(x)h''(x) - h'''(x)\psi_1\psi_2)\delta x - h''(x)(\delta\psi_1)\psi_2 - h''(x)\psi_1(\delta\psi_2)\\ &= (h'(x)h''(x) - h'''(x)\psi_1\psi_2)(\epsilon_1\psi_1 + \epsilon_2\psi_2) - h''(x)(h'(x)\epsilon_2)\psi_2 - h''(x)\psi_1(-h'(x)\epsilon_1)\\ &= h'(x)h''(x)(\epsilon_1\psi_1 + \epsilon_2\psi_2) - h'(x)h''(x)(\epsilon_2\psi_2 - \psi_1\epsilon_1)\\ &= h'(x)h''(x)(\epsilon_1\psi_1 + \epsilon_2\psi_2) - h'(x)h''(x)(\epsilon_2\psi_2 + \epsilon_1\psi_1)\\ &= 0.\end{aligned}$$

Since S is invariant under our supersymmetric transformation, S is called supersymmetric.

There is a localization principle for path integrals: "The path integral becomes localized on the field configurations for which the fermionic variables are invariant under supersymmetry." In the example at hand, the fermionic variables are invariant if $h'(x)\epsilon_2 = -h'(x)\epsilon_1 = 0$ for arbitrary ϵ_i. This is the case when $h'(x) = 0$. Thus, if we assume h is a nonconstant polynomial, then the path integral becomes a finite sum over the critical points of h.

We note here that the discussion in [1] of the path integral on the locus of $h'(x) \neq 0$ is incorrect, and a correction is posted on the AMS webpage for [1]. We will not go through that discussion.

Let x_c be a critical point of h, and assume $h''(x_c) \neq 0$. Near x_c, we may write $h(x) = h(x_c) + \frac{h''(x_c)}{2}(x - x_c)^2$. We ignore higher orders since S does not depend on them. Then the path integral contribution near x_c is

$$\begin{aligned}Z &= \int \exp(-h'(x)^2/2)h''(x)dx\\ &= h''(x_c)\int \exp(-h''(x_c)^2(x - x_c)^2/2)dx\\ &= h''(x_c)\sqrt{\frac{\pi}{h''(x_c)^2}}\\ &= \pm\sqrt{\pi}.\end{aligned}$$

The sign is the same as the sign of $h''(x_c)$. Once again, the $\sqrt{\pi}$ term can be absorbed into the path integral. We get that the total partition function is a sum over the critical points x_c of the sign of $h''(x_c)$. The terms of this sum will cancel to get an integer between -1 and 1, which is what we arrived at previously. However, we would like to avoid the issue of signs, so that we can count the critical points

To do so, we introduce complex variables and their conjugates. In particular, we will treat z and $\bar{z}$ as independent bosons; similarly we will have fermions ψ_1, ψ_2 and $\bar{\psi}_1, \bar{\psi}_2$. Let $W(z)$ be holomorphic in z. The corresponding action is

$$S = |W'(z)|^2 - W''(z)\psi_1\psi_2 - \overline{W''(z)}\bar{\psi}_1\bar{\psi}_2.$$

Infinitesimal transformations will also have corresponding conjugate versions. In particular, for fermionic parameters ϵ_1, ϵ_2, we have a supersymmetric transformation given by

$$\begin{aligned} \delta z &= \epsilon_1\psi_1 + \epsilon_2\psi_2, & \bar{\delta}\bar{z} &= \epsilon_1\bar{\psi}_1 + \epsilon_2\bar{\psi}_2, \\ \delta\psi_1 &= \epsilon_2\overline{W'(z)}, & \bar{\delta}\bar{\psi}_1 &= \epsilon_2 W'(z), \\ \delta\psi_2 &= -\epsilon_1\overline{W'(z)}, & \bar{\delta}\bar{\psi}_2 &= -\epsilon_1 W'(z). \end{aligned}$$

All variations not explicitly listed, such as $\delta\bar{z}$, are 0. We show directly that $\delta S = \bar{\delta}S = 0$. We have

$$\begin{aligned} \delta S &= W''(z)\delta z\overline{W'(z)} - W'''(z)\delta z\psi_1\psi_2 - W''(z)\delta\psi_1\psi_2 - W''(z)\psi_1\delta\psi_2 \\ &= W''(z)\overline{W'(z)}(\epsilon_1\psi_1 + \epsilon_2\psi_2) - W''(z)\overline{W'(z)}\epsilon_2\psi_2 + W''(z)\overline{W'(z)}\psi_1\epsilon_1 \\ &= 0, \\ \bar{\delta}S &= W'(z)\overline{W''(z)}\bar{\delta}\bar{z} - \overline{W'''(z)}\bar{\delta}\bar{z}\bar{\psi}_1\bar{\psi}_2 - \overline{W''(z)}\bar{\delta}\bar{\psi}_1\bar{\psi}_2 - \overline{W''(z)}\bar{\psi}_1\bar{\delta}\bar{\psi}_2 \\ &= W'(z)\overline{W''(z)}(\epsilon_1\bar{\psi}_1 + \epsilon_2\bar{\psi}_2) - W'(z)\overline{W''(z)}\epsilon_2\bar{\psi}_2 + W'(z)\overline{W''(z)}\bar{\psi}_1\epsilon_1 \\ &= 0. \end{aligned}$$

We have used $|z|^2 = z\bar{z}$, and that a holomorphic function in z has no dependence on $\bar{z}$. Next, we compute the partition function for this action. First, we integrate the fermionic variables. Note that

$$e^{-S} = e^{-|W'|^2}(1 + W''\psi_1\psi_2)(1 + \overline{W''}\bar{\psi}_1\bar{\psi}_2),$$

so that

$$Z = \int |W''|^2 e^{-|W'|^2} dz d\bar{z}.$$

Once again, we write $W(z) = W(z_c) + \frac{W''(z_c)}{2}(z - z_c)^2$ near a critical point of W. As a short-hand, we write W''_c for $W''(z_c)$. The contribution to the path integral is then

$$|W''_c|^2 \int e^{-|W''_c|^2|z-z_c|^2} dz d\bar{z} = |W''_c|^2 \int e^{-|W''_c|^2(x^2+y^2)}(-2idxdy) = -2\pi i.$$

The contribution is the same at each critical point, so we can normalize by this amount to find that the partition function counts the critical points of W.

References

[1] Katz, S., *Enumerative Geometry and String Theory*, American Mathematical Society, 2006.

[2] Bourque, A., Zeitlin, A., *Flat* $GL(1|1)$*-connections and fatgraphs*, Journal of Geometry and Physics, 2023.