Points-Lines-Planes Conjecture

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Abstract

In this paper, we give a proof of a special case of the points-lines-planes conjecture, which in turn is a special case of a conjecture which says that the Whitney numbers of the second kind are log-concave.

1 Introduction

Given a matroid M, the Whitney numbers of the second kind[1], denoted W_k, are defined as the number of rank k flats of M. The Whitney numbers have long been presumed to have various interesting properties. One such property is log-concavity, and it is still an open conjecture as to whether this property holds for every matroid. Other sequences of numbers associated to a matroid have been shown to be log-concave. The proofs of these seemingly simple discrete statements involved advanced machinery from fields such as algebraic geometry and Hodge theory.

One conjecture about the Whitney numbers which was recently proved is the so-called Top-Heavy conjecture of Dowling and Wilson ([DW74, DW75]). This states that, for $i < r/2$, where r is the rank of M, we have $W_i \leq W_{r-i}$ and $W_i \leq W_{i+1}$. This was proved by Huh and Wang in 2017 ([HW17]) for realizable matroids, and then it was proved for all matroids by Braden, Huh, Matherne, Proudfoot, and Wang in 2020 ([BHM+20]). It is a classic result of de Bruijn and Erdös (see page 64 of [AZ10]) that for $n \geq 3$ points in the plane, not all of which are on a line, there are at least n lines which pass through at least two of the points. This is a special case of the Top-Heavy conjecture, as it says that $W_1 \leq W_2$ for matroids of rank at least 3.

In this paper, we give a proof of a special case of log-concavity of the Whitney numbers. The proof presented in this paper is essentially that of Seymour (Section 2 of [Sey82]), but with more details as necessary.

[1] For brevity, we will not say "of the second kind."

2 Preliminaries

2.1 Matroids

We will begin with a review of the definitions of matroids and their flats.

Definition 1. A **matroid** is a pair $M = (E, I)$, where E is a finite set and I is a set of subsets of E satisfying the following axioms:

1. $\varnothing \in I$.
2. If $A' \subseteq A \subseteq E$ and $A \in I$, then $A' \in I$.
3. If $A, B \in I$ and $|A| > |B|$, then there exists an $x \in A$ such that $x \notin B$ and $B \cup \{x\} \in I$.

Elements of I are called the **independent sets** of M.

Example 2. A *realizable matroid* is[2] one which is formed by taking E to be a finite subset of a vector space and taking I to be those subsets of E which consist of linearly independent vectors.

Definition 3. Given a matroid $M = (E, I)$ and a set $A \subseteq E$, the **rank** of A is defined to be the largest n such that there exists $B \in I$ with $|B| = n$ and $B \subseteq A$. The **rank of the matroid** is defined to be the rank of E.

Definition 4. Given a matroid $M = (E, I)$, a set $A \subseteq E$ is a **flat** if for every $x \in E$ with $x \notin A$, the rank of $A \cup \{x\}$ is strictly greater than the rank of A.

Definition 5. A matroid $M = (E, I)$ is **simple** if the rank 1 flats of M are exactly the singleton subsets of E.

Definition 6. Let $A \subset E$. One can form the set

$$\hat{A} = A \cup \{x \in E \mid x \notin A, \text{rank}(A \cup \{x\}) = \text{rank}(A)\},$$

which is a flat. We call this the **flat-closure** of A. If A is a flat, then $A = \hat{A}$.

Definition 7. Given a matroid $M = (E, I)$ of rank r and $i \in \{0, ..., r\}$, we let W_i be the number of rank i flats. W_i is the ith **Whitney number**.

Lemma 8. *If matroid $M = (E, I)$ has rank r and $i \in \{0, ..., r\}$, then $W_i \neq 0$.*

Proof. Note that $W_0 = W_r = 1$, since $\varnothing$ is the only flat (in fact, the only set) of rank 0, and E is the only flat of rank r. This is enough for $r \in \{0, 1\}$. Assume that $r > 1$ and let $i \in \{1, ..., r-1\}$. Since E has rank r, there is a set A of size r in I. Any subset of A is in I by the axioms of a matroid, so there is some set B of size i in I. Its flat-closure $\hat{B}$ is then a flat of rank i, since taking flat-closure does not change rank by definition, and B has rank i. Thus there is at least one flat of rank i, i.e. $W_i \neq 0$. □

[2]Technically, a realizable matroid is *isomorphic* to a matroid described by vectors.

2.1.1 Notation and Assumptions on Matroids

Unless otherwise stated, any given matroid will be simple. Let M be a matroid. A **point** of M is a rank 1 flat. Similarly, a **line** is a rank 2 flat, and a **plane** is a rank 3 flat. Let T be the set of points, let L be the set of lines, and let P denote the set of planes. By definition, $W_1 = |T|, W_2 = |L|, W_3 = |P|$.

Any variable denoted by e, e', e_1, etc. will be assumed to be an element of T. Similarly, variables l and p will be assumed to be elements of L and P respectively. We will also use short-hands in sums, such as $\sum_e f(e)$ in place of $\sum_{e \in T} f(e)$. If X is a flat of rank k, we let $d(X)$ be the number of rank $k-1$ flats contained in X. For $e \in T$ and $l \in L$, define $\phi(e)$ and $\psi(l)$ to be

$$\phi(e) = \sum_{l \supset e} \frac{1}{d(l)}, \qquad \psi(l) = \sum_{p \supset l} \frac{1}{d(p)}.$$

2.2 Hall's Marriage Theorem

We will need an important result from graph theory, known as Hall's marriage theorem. We give the necessary terminology to state it.

Definition 9. A **graph** G is a pair (V, E) where V is a set, called the **vertex set**, and E is a set of 2-element subsets of V, called the **edge set**. Elements of V are called **vertices**, and elements of E are called **edges**. If $\{x, y\} \in E$, we say there is an **edge between** x and y. Two edges **share a vertex** if they are not disjoint. A graph is **finite** if V is a finite set.

Definition 10. A graph $G = (V, E)$ is **bipartite** if the vertex set V can be written as a disjoint union of subsets X and Y, such that there is no edge between two elements of X, and there is no edge between two elements of Y. If this is the case, we write $G = (X, Y, E)$.

Definition 11. Given a bipartite graph $G = (X, Y, E)$, an **X-perfect matching** $\mathcal{M}$ in G is a set of $|X|$ distinct edges of G, such that no two edges in $\mathcal{M}$ share a vertex.

Definition 12. Given a graph $G = (V, E)$ and a vertex v, we define the **neighborhood** $N_G(v)$ to be the set of vertices w such that $\{v, w\} \in E$. More generally, for a subset W of V, we define $N_G(W)$ to be the union of the $N_G(v)$ for all $v \in W$.

Theorem 13 (Hall's Marriage Theorem)**.** *Let $G = (X, Y, E)$ be a finite bipartite graph. Then there exists an X-perfect matching if and only if for every $W \subseteq X$, we have $|W| \leq |N_G(W)|$.*

3 The Main Theorem

Theorem 14. *Let M be a matroid. If no line of M contains more than three points, then $W_2^2 \geq W_1 W_3$.*

Proof. Our proof will be split into several lemmas.

Lemma 15. *$W_2^2 \geq W_1 W_3$ if and only if $0 \leq W_2 \sum_e \phi(e) - W_1 \sum_l \psi(l)$.*

Proof. We show that $\sum_e \phi(e) = W_2$ and $\sum_l \psi(l) = W_3$, so that the second condition says $0 \leq W_2^2 - W_1 W_3$, which is clearly equivalent to the first condition. We have

$$\sum_e \phi(e) = \sum_e \sum_{l \supset e} \frac{1}{d(l)} = \sum_l \sum_{e \subset l} \frac{1}{d(l)} = \sum_l 1 = |L| = W_2.$$

Here, we have swapped the order of summation, which is always justified for finite sums. The sum $\sum\limits_{e \subset l} \dfrac{1}{d(l)}$ equals 1 because, by definition, there are $d(l)$ points e satisfying $e \subset l$. A similar calculation shows that $\sum_l \psi(l) = W_3$. □

Lemma 16. *If $e \subset l$, then $\phi(e) \geq \psi(l)$.*

Proof. For each plane p containing l, choose a point $e_p \subset p$ which is not on l. Note that if such a choice was not possible, then we would have $p = l$, which contradicts the assumption that p and l have different ranks. Let l_p be the flat closure of $e \cup e_p$. Then l_p is a line with $e \subset l_p \subset p$ and $l_p \neq l$. For planes $p, p' \supset l$, p is the flat closure of $l \cup l_p$ and p' is the flat closure of $l \cup l_{p'}$, so if $p \neq p'$, then $l_p \neq l_{p'}$. Thus, there are at least as many lines containing e as there are planes containing l.

Now, for $p \supset l$, we have $d(p) \geq d(l_p)$: there is always a point $e' \subset p$ for which $e' \not\subset l_p$, and for each point $e^i \subset l_p$, the flat closure of $e' \cup e^i$ is a line l^i contained in p, with $l^i \neq l^j$ if $e^i \neq e^j$. Using this inequality, we get

$$\psi(l) = \sum_{p \supset l} \frac{1}{d(p)} \leq \sum_{p \supset l} \frac{1}{d(l_p)} \leq \sum_{l' \supset e} \frac{1}{d(l')} = \phi(e),$$

where the second inequality follows because each l_p is a distinct line containing e, so that the sum $\sum\limits_{l' \supset e} \dfrac{1}{d(l')}$ equals the sum $\sum\limits_{p \supset l} \dfrac{1}{d(l_p)}$ plus some extra non-negative terms. □

So far, we have not used the assumption that M has no lines which contain more than 3 points.

Lemma 17. *Suppose no line of M contains more than three points. For any collection of points X, there are at least $|X| W_2 / W_1$ lines which contain a point in X.*

Proof. For brevity, we will say a line intersects X if it contains a point in X. Let $L' \subseteq L$ be the set of lines which intersect X.

We claim that any line l in L' contains at most two pairs (x, y), where $x \in X$ and $y \in T - X$. Indeed, suppose we had three such pairs (x_i, y_i) for $i \in \{1, 2, 3\}$. Since $x_1 \in X$ and $y_1 \notin X$, the points x_1, y_1 are distinct. Similarly, x_2, y_2 are distinct. By assumption, l cannot contain four distinct points. Thus $x_2 = x_1$ or $y_2 = y_1$. If $(x_1, y_1) \neq (x_2, y_2)$, then the third pair (x_3, y_3) is forced to be equal to one of the other two pairs.

Each pair $(x, y) \in X \times (T - X)$ is in a unique member of L', namely the flat-closure of $x \cup y$. There are $|X|(W_1 - |X|)$ of these pairs, so by the previous observation, $|L'| \geq \frac{1}{2}|X|(W_1 - |X|)$.

Any line not in L' is determined by two points not in X, so

$$|L - L'| \leq \binom{W_1 - |X|}{2} = \frac{1}{2}(W_1 - |X|)(W_1 - |X| - 1) \leq \frac{1}{2}(W_1 - |X|)^2.$$

Thus

$$|L'| \geq \frac{1}{2}|X|(W_1 - |X|) = \frac{|X|}{W_1 - |X|}\frac{1}{2}(W_1 - |X|)^2 \geq \frac{|X|}{W_1 - |X|}(W_2 - |L'|),$$

which simplifies to $|L'| \geq |X|W_2/W_1$. □

The conclusion of Lemma 17 allow us to construct a function $\alpha : T \times L \to \mathbb{Z}_{\geq 0}$ with the following properties:

$$\alpha(e, l) = 0 \text{ if } e \not\subset l; \quad \text{For all } e, \sum_l \alpha(e, l) = W_2; \quad \text{For all } l, \sum_e \alpha(e, l) = W_1.$$

The proof of the existence of α is deferred to Section 4.

We can now prove Theorem 14. For any pair $(e, l) \in T \times L$, we know that $\alpha(e, l) \geq 0$. If $e \subset l$, then $\phi(e) - \psi(l) \geq 0$ by Lemma 16. If $e \not\subset l$, then $\alpha(e, l) = 0$, so that $\alpha(e, l)(\phi(e) - \psi(l)) = 0$. Thus, we have that $0 \leq \sum_{e,l} \alpha(e, l)(\phi(e) - \psi(l))$. But

$$\sum_{e,l} \alpha(e, l)\phi(e) = \sum_e \phi(e) \sum_l \alpha(e, l) = W_2 \sum_e \phi(e).$$

Similarly, $\sum_{e,l} \alpha(e, l)\psi(l) = W_1 \sum_l \psi(l)$. Thus $0 \leq W_2 \sum_e \phi(e) - W_1 \sum_l \psi(l)$. By Lemma 15, we have $W_2^2 \geq W_1 W_3$ as desired. □

4 Existence of α

Fix a matroid $M = (E, I)$. Let $m = W_1, n = W_2$. Assume that if $X \subseteq T$, then X intersects at least $|X|n/m$ lines. Let $G = (T, L, D)$ be the bipartite graph where $e \in T$ and $l \in L$ are connected by an edge if and only if $e \subset l$. Then, consider the bipartite graph $\hat{G} = (\hat{T}, \hat{L}, \hat{D})$, constructed as follows.[3] Write $T = \{e_1, ..., e_m\}, L = \{l_1, ..., l_n\}$. Let $\hat{e}_i$ be an n element set, say $\hat{e}_i = \{x_{i,1}, ..., x_{i,n}\}$. Similarly, let $\hat{l}_i = \{y_{i,1}, ..., y_{i,m}\}$. Furthermore, suppose that for $i \neq j$, $\hat{e}_i \cap \hat{e}_j = \varnothing = \hat{l}_i \cap \hat{l}_j$. Then let $\hat{T} = \bigcup_{i=1}^{m} \hat{e}_i$ and $\hat{L} = \bigcup_{i=1}^{n} \hat{l}_i$. Vertices $x_{i,j} \in \hat{T}$ and $y_{s,t} \in \hat{L}$ are connected by an edge if and only if $e_i \subset l_s$.

An informal yet more digestible way to describe the construction of $\hat{G}$ is as follows. From the graph G, replace each vertex in T by W_2 copies of itself, and replace each vertex in L by W_1 copies of itself. An edge in G between e and l is replaced by $W_1 W_2$ edges between the copies of e and l.

Let $X \subset \hat{T}$. For $i \in \{1, \ldots, m\}$, let r_i be the number of elements in X that are of the form $x_{i,t}$ for $t \in \{1, ..., n\}$. Then $r_i \leq n$. We can relabel the points of the matroid so that there is some number j such that $r_i = 0$ if and only if $i > j$. Then $|X| = r_1 + ... + r_j \leq jn$.

Now, suppose there are k lines which intersect $\{e_1, ..., e_j\}$. Then $|N_{\hat{G}}(X)| = km$, since each of the k lines corresponds to m vertices in $\hat{G}$. Furthermore, by the assumption on M, we have $k \geq jn/m$. Then

$$|N_{\hat{G}}(X)| = km \geq jn \geq |X|.$$

By Hall's marriage theorem, there exists a $\hat{T}$-perfect matching $\mathcal{M}$ in $\hat{G}$. Since $|\hat{T}| = mn = |\hat{L}|$, $\mathcal{M}$ is also an $\hat{L}$-perfect matching. We define $\alpha(e, l) = |\{(x, y) \in \mathcal{M} : x \in \hat{e}, y \in \hat{l}\}|$. By construction, there is only an edge between a vertex in $\hat{e}$ and a vertex in $\hat{l}$ if $e \subset l$; thus $\alpha(e, l) = 0$ if $e \not\subset l$. For a fixed point e, the sum $\sum_l \alpha(e, l)$ equals the size of $\{(x, y) \in \mathcal{M} : x \in \hat{e}\}$. By the definition of a perfect matching, the size of this set is $|\hat{e}| = n = W_2$. Similarly, for a fixed line l, we have $\sum_e \alpha(e, l) = W_1$. This α therefore satisfies the desired criteria needed in the proof of Theorem 14.

[3] I owe this construction, and particularly its use in this proof, to Antoine Labelle.

References

[AZ10] M. Aigner and G. M. Ziegler. *Proofs from The Book*. Springer, Berlin, sixth edition, 2010.

[BHM+20] T. Braden, J. Huh, J. P. Matherne, N. Proudfoot, and B. Wang. Singular Hodge theory for combinatorial geometries. *preprint*, arXiv:2010.06088, 2020.

[DW74] T. A. Dowling and R. M. Wilson. The slimmest geometric lattices. *Trans. Amer. Math. Soc.*, 196:203–215, 1974.

[DW75] T. A. Dowling and R. M. Wilson. Whitney number inequalities for geometric lattices. *Proc. Amer. Math. Soc.*, 47:504–512, 1975.

[HW17] J. Huh and B. Wang. Enumeration of points, lines, planes, etc. *Acta Math.*, 218(2):297–317, 2017.

[Sey82] P. D. Seymour. On the points-lines-planes conjecture. *J. Combinatorial Theory Ser. B*, 33(1):17–26, 1982.