A GENTLE EXPOSITION TO DIFFERENTIAL FORMS

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ABSTRACT. This paper is a short, hands-on summary of key ideas in differential forms. In particular, we will show how they can be used as a tool in algebraic topology. However, differential forms are also useful in various other branches of mathematics, such as differential geometry, algebraic geometry, and mathematical physics, to name a few. They also provide generalizations of familiar facts from calculus and physics courses.

1. INTRODUCTION

In a standard graduate topology course, the first algebraic invariant of a topological space introduced is usually the fundamental group. In particular, one can say that two spaces are not homeomorphic ("the same") if they have non-isomorphic ("different") fundamental groups. However, non-homeomorphic spaces can have the same fundamental group, so it helps to have develop even more ways to distinguish spaces algebraically. The next algebraic invariants one learns (perhaps in a second course on topology) are the homology groups. The typical intuition provided for these groups are that they count the "number of holes" in a topological space. Finally, one learns of cohomology, which is much harder to provide intuition for. In many examples, it seems that the homology groups give the same information as the cohomology groups. However, the cohomology groups of a space have the extra benefit of coming together to form a ring, and this ring structure can be used to distinguish topological spaces that have the same cohomology groups. In particular, cohomology is a very strong invariant of a topological space.

The main goal of this paper is to introduce differential forms as a tool which can compute the cohomology of certain topological spaces, namely (smooth) manifolds. More precisely, we study the de Rham cohomology, which agrees with the usual topological approach to cohomology in most "nice" cases. Differential forms are perhaps the simplest way to develop cohomology, since most of the computations involve nothing more than familiar calculus techniques.

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One key result in the subject of differential forms is the (generalized) Stokes' Theorem, which gives a simultaneous generalization of the Fundamental Theorem of Calculus, Green's Theorem, and the Divergence Theorem. These theorems, which are encountered and used to solve countless problems in calculus and physics classes, can be seen as special cases of a beautiful statement in the language of differential forms.

We also mention, for the physics-inclined reader, that there is an interesting (and short!) formulation of Maxwell's equations in the language of differential forms. However, this will not be explored in this paper.

2. Manifolds

Differential forms live on certain topological spaces called smooth manifolds. Thus, before we can even give an intuitive idea of what differential forms are, we need to give the appropriate background.

The idea of a manifold is actually already familiar from calculus. In calculus, the notions of derivative and linearization (or first order approximation) come from the idea that curves look like lines close up. In topology, this idea is applied to an arbitrary dimension n. That is to say, for some fixed n, an n dimensional manifold is a topological spaces which "looks like" $\mathbb{R}^n$. The precise meaning of "look like" can depend on the type of manifolds one considers – topological, differentiable, smooth, analytic, and so on. For the setting of differential forms, we will only consider *smooth* manifolds, and henceforth drop the word "smooth" and just say "manifold". Let us now give the precise definition of a manifold.

Definition 2.1. A *smooth n-dimensional manifold* (or n-manifold for short) is a topological space M together with a collection of pairs (U_α, ϕ_α) called *charts*, such that:

(1) each U_α is an open subset of M,
(2) the union of all the sets U_α is equal to M,
(3) each ϕ_α is a homeomorphism from U_α to $\mathbb{R}^n$,
(4) and the *transition functions* $\phi_\alpha \circ \phi_\beta^{-1}$ restricted to $\phi_\beta(U_\alpha \cap U_\beta)$ are infinitely differentiable.

In more digestible words, conditions (1), (2), and (3) formalize the idea that "a manifold is covered by pieces which look like $\mathbb{R}^n$," and condition (4) formalizes the idea that each $\mathbb{R}^n$ piece on a manifold is compatible with any other piece it intersects. Put another way, manifolds can be thought of as gluing copies of $\mathbb{R}^n$ together in a smooth and compatible way.

A tautological example of an n-manifold is $\mathbb{R}^n$, equipped with a single chart $(\mathbb{R}^n, \mathrm{id})$. A more interesting example, while still familiar, is the n-sphere S^n. For instance, S^1 is a circle. If we fix a particular realization of the circle, say as the set of points (x, y) in $\mathbb{R}^2$ satisfying $x^2 + y^2 = 1$ equipped with the subspace topology, then we can give S^1 the data of a smooth manifold with two charts $(U_1, \phi_1), (U_2, \phi_2)$, as follows.[1] Let $U_1 = \{(x, y) \in S^1 \mid y \neq 1\}$ and let $U_2 = \{(x, y \in S^1 \mid y \neq -1\}$. Let $\phi_1(x, y) = \dfrac{x}{1-y}$ and $\phi_2(x, y) = \dfrac{x}{1+y}$. One can then check that the four conditions of Definition 2.1 are satisfied.

It should be noted that there is not a unique way to equip a given topological space with a manifold structure. When we refer to a manifold, we refer to both the underlying topological space *and* a given collection of charts.

2.1. **Local Coordinates.** Recall that the topological space $\mathbb{R}^n$ is the product of n copies of the topological space $\mathbb{R}$. It comes with n continuous maps $u_i : \mathbb{R}^n \to \mathbb{R}$ which map a point $p = (p_1, ..., p_n)$ in $\mathbb{R}^n$ to p_i. These maps u_i can be thought of as providing a coordinate system for $\mathbb{R}^n$. We want something similar for manifolds.

It turns out that we cannot (always) describe points on manifolds uniquely by coordinates. However, we can give a manifold a *local* coordinate system. In particular, each chart (U_α, ϕ_α) of an n-manifold gives rise to coordinate functions $x_i = u_i \circ \phi_\alpha : U_\alpha \to \mathbb{R}$ for $i \in \{1, ..., n\}$. In other words, a point $p \in U_\alpha$ can be associated to the coordinates $(p_1, ..., p_n)$ with each $p_i \in \mathbb{R}$, where $(p_1, ..., p_n)$ are the coordinates of $\phi_\alpha(p)$ in $\mathbb{R}^n$.

One caveat of local coordinates is the following. A point p on a manifold M can belong to two subsets U_α and U_β, and therefore be given two different coordinates. For example, the point $\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ on S^1 can be associated to the coordinate $\sqrt{2} + 1$ via ϕ_1, and it can also be associated to the coordinate $\sqrt{2} - 1$ via ϕ_2. One can verify that $\phi_1(x, y)\phi_2(x, y) = 1$ if and only if $(x, y) \in U_1 \cap U_2$. In other words, for any point in the overlap $U_1 \cap U_2$, the two coordinates associated to it are reciprocals of each other.

In general, since the maps ϕ_α and ϕ_β are homeomorphisms, there is a bijection between coordinates of points in $U_\alpha \cap U_\beta$ given by ϕ_α,

[1] This procedure is known as stereographic projection, and can be used analogously for S^n in general.

and the coordinates of such points given by ϕ_β. However, a bijection may not be strong enough. It is exactly condition (4) of Definition 2.1 that ensures we can switch between coordinate systems in a way that would not "break" any sort of calculus ideas that we want to employ on manifolds.

From here on out, things will only get more technical. However, the one thing to keep in mind is the following general pattern of manifold theory:

(1) Locality: do/define something with local coordinates.
(2) Compatibility: check that the computation/construction is well-defined with respect to a change of coordinate systems.
(3) Gluing: conclude that the computation/construction holds for the whole manifold.

For brevity, we will mainly focus on "locality", as the other two (especially gluing) are more technical.

2.2. **Functions.** A common theme in mathematics, especially in algebra, geometry, and topology, is that we do not just study an object on its own. We are almost always interested in the ways two objects of the same type are related. For instance, in algebra, we study groups *as well as* homomorphisms between two groups. Similarly, in topology, we study topological spaces and continuous maps between two spaces. The corresponding notion in our setting is that of smooth maps. We will not need the full generality of maps between arbitrary manifolds; instead, we only need maps from a manifold to $\mathbb{R}^m$.

Definition 2.2. Let M be an n-manifold, and let $f : M \to \mathbb{R}^m$ be a continuous function. We say f is *smooth* if, for each chart (U_α, ϕ_α) of M, the function $f \circ \phi_\alpha^{-1} : \mathbb{R}^n \to \mathbb{R}^m$ is infinitely differentiable.

Since $\mathbb{R}^m$ is a vector space, we can add two functions $f, g : M \to \mathbb{R}^m$ together, *pointwise*. Namely, we define $f+g : M \to \mathbb{R}^m$ by $(f+g)(p) = f(p)+g(p)$. Similarly, for $c \in \mathbb{R}$, we can multiply a function f by c. One can check that the operations of addition and scalar multiplication take smooth functions to smooth functions. In particular (skipping some other details), the smooth functions $M \to \mathbb{R}^m$ form a vector space over $\mathbb{R}$, which we denote by $\mathcal{C}^\infty(M, \mathbb{R}^m)$.

Now consider a smooth map $f : M \to \mathbb{R}^m$. Let $p \in M$ be a point, and let (U_α, ϕ_α) be a chart such that $p \in U_\alpha$. Also let (x_i) be local coordinates on U_α. Then we define

$$\frac{\partial f}{\partial x_i}(p) = \frac{\partial (f \circ \phi_\alpha^{-1})}{\partial u_i}(\phi_\alpha(p)).$$

The motivation for this definition is that derivatives are local operations. To take the derivative of a function at a point on a n-manifold, one can zoom in close enough and treat said function as a function on $\mathbb{R}^n$. Note that the notation on the left hand side does not mention the chart. Indeed, the computation of partial derivatives is independent of the choice of chart.

If we just write $\dfrac{\partial f}{\partial x_i}$, we mean the function $M \to \mathbb{R}^m$ which sends $p \in M$ to $\dfrac{\partial f}{\partial x_i}(p)$. This function is also smooth.

As an example, consider the map $f : S^1 \to \mathbb{R}$ which sends $(x, y) \in S^1$ to $x \in \mathbb{R}$. We claim this is smooth. For instance, the function $f \circ \phi_1^{-1}$ sends a real number t to $\dfrac{t^2}{t^2+1}$, and one can check that this is an infinitely differentiable function. We invite the reader to compute $f \circ \phi_2^{-1}$ and check that it is also infinitely differentiable. In the chart (U_1, ϕ_1), we compute $\dfrac{\partial f}{\partial x_1}\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ to be

$$\frac{\partial}{\partial u_1}\bigg|_{u_1=\sqrt{2}+1}\left(\frac{u_1^2}{u_1^2+1}\right) = \frac{1}{4}(\sqrt{2}-1).$$

As another exercise for the reader, compute $\dfrac{\partial f}{\partial x_1}\left(\dfrac{1}{\sqrt{2}}, \dfrac{1}{\sqrt{2}}\right)$ in the chart (U_2, ϕ_2). Both computations lead to the same number, indicating that the definition of the derivative is independent of the choice of chart.

2.3. **Tangent Space.** In this subsection, we will introduce the last piece of our setting: the tangent space to a point of a manifold. Intuitively, one can picture the numerous tangent lines to curves drawn in a calculus class. However, a point on the tangent space corresponds to something one may not expect: a differential operator.

Definition 2.3. Let M be an n-manifold, let p be a point on M, and let $x_1, ..., x_n$ be a local coordinate system coming from a chart containing p. The *tangent space* to M at p, denoted T_pM, is the vector space over $\mathbb{R}$ spanned by the functions (or operators) $\dfrac{\partial}{\partial x_1}(p), ..., \dfrac{\partial}{\partial x_n}(p)$: $\mathcal{C}^\infty(M, \mathbb{R}^m) \to \mathbb{R}^m$. Elements of T_pM are called *tangent vectors.*

In other words, an element of T_pM is an operator $\displaystyle\sum_{i=1}^n a_i \frac{\partial}{\partial x_i}(p)$, which takes in a smooth function $f : M \to \mathbb{R}^m$ and spits out an element of

$\mathbb{R}^m$, namely $\displaystyle\sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i}(p)$. One can intuitively think of a tangent vector as a directional arrow sitting at a point p, and this arrow tells a function how it will change if it moves away from p in the direction of that arrow.

3. Differential Forms

A differential form can be understood intuitively as an object which lives on a manifold, and at each point of the manifold, takes in a certain number of elements of the tangent space and gives back a real number. In more fun terms, they "eat" tangent vectors. The number of tangent vectors, k, a differential form "eats" does not change from point to point. We call a differential form that "eats" k vectors a *k-form.*

Let us first define differential forms on $\mathbb{R}^n$.

Definition 3.1. Consider $\mathbb{R}^n$ with its standard coordinate system $u_1, ..., u_n$. We say that the $\mathbb{R}$-vector space of *0-forms* on $\mathbb{R}^n$, denoted $\Omega^0(\mathbb{R}^n)$, is $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. For $k \in \{1, ..., n\}$, the $\mathbb{R}$-vector space of *k-forms* on $\mathbb{R}^n$, denoted $\Omega^k(\mathbb{R}^n)$, is the set of formal expressions

$$\sum_{1 \leq i_1 < ... < i_k \leq n} f_{i_1 ... i_k} du_{i_1} \cdots du_{i_k},$$

where the $f_{i_1 ... i_k}$ are elements of $\mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R})$. Given a k-form ω (in the exact same notation as above), a point p in $\mathbb{R}^n$, and k tangent vectors $v_i = \displaystyle\sum_{j=1}^{n} a_{i,j} \frac{\partial}{\partial u_j}(p) \in T_p\mathbb{R}^n$ for $i \in \{1, ..., k\}$, define $\omega(p)(v_1, ..., v_k)$ to be

$$\sum_{1 \leq i_1 < ... < i_k \leq n} f_{i_1 ... i_k}(p) a_{1,i_1} \cdots a_{k,i_k}.$$

In particular, $du_i(p)\left(\dfrac{\partial}{\partial u_j}(p)\right)$ is 1 if and only if $i = j$, and 0 otherwise. If $k > n$, we define $\Omega^k(\mathbb{R}^n) = \{0\}$.

The length of this definition should not deter the reader. Remember the intuition: forms "eat" tangent vectors. Here are some example computations:

$$\omega = u_2 du_1 + u_1 du_2 \in \Omega^1(\mathbb{R}^2), p = (1,2) \in \mathbb{R}^2, v_1 = -5\frac{\partial}{\partial u_1}(p):$$
$$\omega(p)(v_1) = (u_2 du_1 + u_1 du_2)(1,2)\left(-5\frac{\partial}{\partial u_1}(1,2)\right)$$
$$= u_2(1,2)\cdot(-5) = -10.$$
$$\omega = u_3 du_1 du_2, p = (1,0,-1), v_1 = \pi\frac{\partial}{\partial u_1}(p) + e\frac{\partial}{\partial u_3}(p), v_2 = v_1:$$
$$\omega(p)(v_1, v_2) = -1\cdot\pi\cdot 0 = 0.$$

Returning to theory, we still must give a definition of a differential form on a manifold, not just on $\mathbb{R}^n$. However, knowing what a differential form on $\mathbb{R}^n$ is gives us almost the entire picture. More precisely, we know what a differential form should be on any given chart of a manifold. The full definition is as follows:

Definition 3.2. Given a manifold M with charts (U_α, ϕ_α), a differential form ω on M is a collection of differential forms ω_α on $\mathbb{R}^n$, one for each chart of M, with a certain compatibility condition.

For simplicity, we will not worry ourselves with the compatibility condition. As we progress, our computations and definitions will be on a fixed chart of a manifold, with local coordinates (x_i). In accordance, the notation of a differential form will use symbols dx_i instead of du_i.

3.1. **Exterior Derivative.** So far, given a manifold M, we have a sequence of vector spaces $\Omega^k(M)$. We now want to put these vector spaces together into something called a *chain complex.* This involves giving linear maps $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$ such that $d_{k+1} \circ d_k : \Omega^k(M) \to \Omega^{k+2}(M)$ is the map which sends anything to 0. In practice, authors may forgo the subscript and simply write $d : \Omega^k(M) \to \Omega^{k+1}(M)$, but we will keep the subscript for clarity.

Before we can define these maps, we need to say something about the ordering of the symbols dx_i. In Definition 3.1, we defined a k-form so that the symbols dx_i are given in order. But what about something like $dx_2 dx_1$? We impose the condition that exchanging two symbols dx_i and dx_j comes at the cost of a minus sign: $dx_i dx_j = -dx_j dx_i$. Furthermore, we impose the condition that $dx_i dx_i = 0$. Intuitively, one can think of the first condition as giving orientation to differential forms; this idea is important for defining integration of forms. The second condition mirrors how we treat “infinitesimals” in first order approximation.

Let us first define $d_0 : \Omega^0(M) \to \Omega^1(M)$. An element of $\Omega^0(M)$, i.e. a 0-form, is equivalent to the data of a smooth function $f : M \to \mathbb{R}$. We define

$$d_0(f) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i.$$

One can check that this map is linear. Next, given $\omega \in \Omega^k(M)$, say $\omega = \sum_{1 \leq i_1 < \ldots < i_k \leq n} f_{i_1 \ldots i_k} dx_{i_1} \cdots dx_{i_k}$ in a chart, we define

$$d_k(\omega) = \sum_{1 \leq i_1 < \ldots < i_k \leq n} d_0(f_{i_1 \ldots i_k}) dx_{i_1} \cdots dx_{i_k}.$$

This map is also linear.

As an example computation, consider the 1-form $x_2 dx_1$ on $\mathbb{R}^2$. Then $d_1(x_2 dx_1) = d_0(x_2) dx_1 = dx_2 dx_1 = -dx_1 dx_2$. Similarly, $d_1(x_1^2 dx_1) = 2x_1 dx_1 dx_1 = 0$.

Each map d_k is referred to as the *exterior derivative*. We mention as an aside that in the case of $\mathbb{R}^3$, the maps d_k are analogous to the gradient, curl, and divergence from multivariable calculus. More details of this are given on page 14 of [1].

In order to define the de Rham cohomology of a manifold, we must verify that $d_{k+1} \circ d_k = 0$, i.e. $d_{k+1} \circ d_k : \Omega^k(M) \to \Omega^{k+2}(M)$ sends any k-form to 0.

Proposition 3.3. *The map $d_{k+1} \circ d_k : \Omega^k(M) \to \Omega^{k+2}(M)$ sends $\omega \in \Omega^k(M)$ to 0.*

Proof. Since the composition of linear maps is linear, we know that $d_{k+1} \circ d_k$ is linear. Thus, it suffices to check that $d_{k+1}(d_k(\omega)) = 0$ for $\omega = f dx_{i_1} \cdots dx_{i_k}$. First, we have $d_k(\omega) = d_0(f) dx_{i_1} \cdots dx_{i_k} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i dx_{i_1} \cdots dx_{i_k}$. Then

$$d_{k+1}(d_k(\omega)) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i dx_{i_1} \cdots dx_{i_k}.$$

For infinitely differentiable functions, the order of partial differentiation does not matter. We also have $dx_j dx_i = -dx_i dx_j$. Since the we are taking a finite sum, we can exchange the order of summation. Thus, we have

$$d_{k+1}(d_k(\omega)) = -\sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j dx_{i_1} \cdots dx_{i_k}.$$

We can swap the names of the indices i and j, so that

$$d_{k+1}(d_k(\omega)) = -\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i} dx_j dx_i dx_{i_1} \cdots dx_{i_k}.$$

Thus $d_{k+1}(d_k(\omega)) = -d_{k+1}(d_k(\omega))$, so $d_{k+1}(d_k(\omega)) = 0$. □

3.2. **de Rham Cohomology.**

Definition 3.4. The vector space of *exact* k-forms, denoted $B^k(M)$, is the image of $d_{k-1} : \Omega^{k-1}(M) \to \Omega^k(M)$. In other words, a k-form ω is exact if there is a $(k-1)$-form τ such that $\omega = d_{k-1}(\tau)$. The vector space of *closed* k-forms, denoted $Z^k(M)$, is the kernel of $d_k : \Omega^k(M) \to \Omega^{k+1}(M)$. In other words, a k-form ω is closed if $d_k(\omega) = 0$.

As a corollary of Proposition 3.3, all exact k-forms are closed. In other words, $B^k(M)$ is a vector subspace of $Z^k(M)$. This allows us to define their quotient.

Definition 3.5. The kth de Rham cohomology vector space of a manifold M, denoted $H^k(M)$, is the vector space $Z^k(M)/B^k(M)$.

As mentioned in the introduction, cohomology has the advantage over homology in that it has a ring structure. In particular, by "ring structure", we mean a way to multiply elements of $H^k(M)$ and $H^\ell(M)$. While the usual topological definition of the cohomology ring is rather complicated, the ring structure for de Rham cohomology is already built into what we have developed.

We can multiply k-forms and ℓ-forms by usual algebraic manipulation; use the distributive property, and simplify by using the exchange rule $dx_i dx_j = -dx_j dx_i$. As an example, consider the 2-form $x_3 dx_1 dx_2$ and the 1-form $x_2 dx_1 + x_1 dx_2$. Then

$$\begin{aligned}(x_3 dx_1 dx_2)(x_2 dx_1 + x_1 dx_2) &= x_2 x_3 dx_1 dx_2 dx_1 + x_1 x_3 dx_1 dx_2 dx_2 \\ &= -x_2 x_3 dx_1 dx_1 dx_2 + x_1 x_3 dx_1 \cdot 0 = -x_2 x_3 \cdot 0 \cdot dx_2 + 0 = 0.\end{aligned}$$

An element of $H^k(M)$ is a coset $[\omega] = \omega + B^k(M)$, represented by some closed k-form ω. Given $[\omega] \in H^k(M)$ and $[\tau] \in H^\ell(M)$, we then define $[\omega][\tau] = [\omega\tau]$. One does need to make sure that this is well-defined: namely that the product of two closed forms is closed, and that the product of classes is independent of the choice of representatives. However, we will not do this. For us, the development of theory ends here.

4. Results

We end this paper with a a few results that one can prove with the machinery we have provided. The interested reader can consult [1] for details.

Proposition 4.1 (Poincaré Lemma). *$H^k(\mathbb{R}^n)$ is 1-dimensional if $k = 0$ and 0-dimensional otherwise.*

This result tells us that cohomology of $\mathbb{R}^n$ is not very interesting. This is why the development of manifolds is relevant. As another example, we have that $H^k(S^n)$ is 1-dimensional is $k = 0$ or $k = n$ and 0-dimensional otherwise.

Proposition 4.2 (de Rham's Theorem). *There is an isomorphism between the de Rham cohomology ring of a manifold and its singular cohomology ring.*

Thus, our calculus-style calculations really do give us classical topological information.

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