MATH 7520 Homework 3

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Problem 1

For X and Y path connected, show that $\pi_n(X \times Y) \cong \pi_n(X) \times \pi_n(Y)$.

Proof. The universal property of products states that a map $f : Z \to X \times Y$ is uniquely determined by two maps $f_1 : Z \to X$ and $f_2 : Z \to Y$. In particular, $f(z) = (f_1(z), f_2(z))$ and $f_i = p_i \circ f$ for $i \in \{1, 2\}$, where p_1 and p_2 are the projections $X \times Y \to X, Y$.

Following this observation (and using the same notation), we define a function $\zeta_n : \pi_n(X \times Y) \to \pi_n(X) \times \pi_n(Y)$ by $\zeta_n([f]) = ([f_1], [f_2])$. We must check several details: that ζ_n is well-defined, that ζ_n is a group homomorphism, and that ζ_n is a bijection.

To start, suppose f, g are homotopic maps $S^n \to X \times Y$, related by a homotopy $h : S^n \times I \to X \times Y$. We claim that for $i \in \{1, 2\}$, the map h_i is a homotopy between f_i and g_i; this claim says that ζ_n is well-defined. h_i is continuous because it is a composition of continuous functions, namely p_i and h. Indeed, we have

$$h_i(s, 0) = p_i(h(s, 0)) = p_i(f(s)) = f_i(s),$$
$$h_i(s, 1) = p_i(h(s, 1)) = p_i(g(s)) = g_i(s).$$

Thus, ζ_n is well-defined.

Now consider a sum (or composition if $n = 1$, but I will stick to using sum and swear to not use abelianicity) of two maps $f, g : S^n \to X \times Y$. We want to show that $\zeta_n([f + g]) = \zeta_n([f]) + \zeta_n([g])$, i.e. $[p_i \circ (f + g)] = [f_i] + [g_i]$ for $i \in \{1, 2\}$. In fact, it suffices to show that $p_i \circ (f + g) = f_i + g_i$. To see this, we expand the definitions (and replace S^n by $I^n/\partial I^n$):

$$(f+g)(s_1,\dots,s_n) = \begin{cases} f(2s_1, s_2, \dots, s_n) & s_1 \le \frac{1}{2} \\ g(2s_1-1, s_2, \dots, s_n) & s_1 \ge \frac{1}{2} \end{cases}$$

$$= \begin{cases} (f_1(2s_1, s_2, \dots, s_n), f_2(2s_1, s_2, \dots, s_n)) & s_1 \le \frac{1}{2} \\ (g_1(2s_1-1, s_2, \dots, s_n), g_2(2s_1-1, s_2, \dots, s_n)) & s_1 \ge \frac{1}{2} \end{cases}$$

$$\therefore (p_i(f+g))(s_1,\dots,s_n) = \begin{cases} f_i(2s_1, s_2, \dots, s_n) & s_1 \le \frac{1}{2} \\ g_i(2s_1-1, s_2, \dots, s_n) & s_1 \ge \frac{1}{2} \end{cases}$$

$$= f_i + g_i.$$

Thus, $[p_i \circ (f+g)] = [f_i + g_i] = [f_i] + [g_i]$ as desired.

Since ζ_n is a homomorphism, we can show that it is injective by showing that it has trivial kernel. To that end, suppose $\zeta_n([f]) = ([c_1],[c_2])$, where each c_i is a constant map; $([c_1],[c_2])$ is the identity in $\pi_n(X) \times \pi_n(Y)$. In particular, we have that $f_i \simeq c_i$ for each i. Let h_i be a homotopy from f_i to c_i. Let h be the map determined by h_1 and h_2, i.e. $h(s,t) = (h_1(s,t), h_2(s,t))$. By the universal property of products, h is continuous. We claim that h is a homotopy between f and the constant map $c(s) = (c_1(s), c_2(s))$. Indeed,

$$h(s,0) = (h_1(s,0), h_2(s,0)) = (f_1(s), f_2(s)) = f(s),$$
$$h(s,1) = (h_1(s,1), h_2(s,1)) = (c_1(s), c_2(s)) = c(s).$$

Thus $[f]$ is the identity in $\pi_n(X \times Y)$, so ζ_n is injective.

Finally, let $([f_1],[f_2]) \in \pi_n(X) \times \pi_n(Y)$. Let f be the map determined by f_1 and f_2. By definition, $\zeta_n([f]) = ([f_1],[f_2])$, so ζ_n is surjective. This concludes the proof that ζ_n is a group isomorphism $\pi_n(X \times Y) \xrightarrow{\sim} \pi_n(X) \times \pi_n(Y)$.

As an aside, we note that the above proof can be modified very slightly to show that $\pi_n(\prod_\alpha X_\alpha) \cong \prod_\alpha \pi_n(X_\alpha)$ for any family of path connected spaces X_α. $\square$

Problem 2

Show that the long exact homotopy sequences of a based pair is natural with respect to continuous maps of based pairs.

Proof. Let $f : (X, A, x_0) \to (Y, B, y_0)$ be a map of based pairs. We take for granted the fact that f induces functions $f_* : \pi_n(X, A, x_0) \to \pi_(Y, B, y_0)$, and that for $n \geq 2$ they are homomorphisms – a claim without proof on Hatcher page 344. For notational clarity, we will use $g : (X, x_0) \to (Y, y_0)$ and $h : (A, x_0) \to (B, y_0)$ for the based maps coming from f. We have the following diagram (which only makes sense as drawn for $n \geq 1$, but the proof of commutativity for each square works when the corresponding square is drawn):

$$\begin{tikzcd}
\pi_n(A, x_0) \arrow[r, "i_*^X"] \arrow[d, "h_*"'] & \pi_n(X, x_0) \arrow[r, "j_*^X"] \arrow[d, "g_*"'] & \pi_n(X, A, x_0) \arrow[r, "\partial^X"] \arrow[d, "f_*"] & \pi_{n-1}(A, x_0) \arrow[d, "h_*"] \\
\pi_n(B, y_0) \arrow[r, "i_*^Y"] & \pi_n(Y, y_0) \arrow[r, "j_*^Y"] & \pi_n(Y, B, y_0) \arrow[r, "\partial^Y"] & \pi_{n-1}(B, y_0)
\end{tikzcd}$$

We wish to show that each square in this diagram commutes. First, let $[k] \in \pi_n(A, x_0)$. Then $g_* i_*^X [k] = [g \circ i^X \circ k]$ and $i_*^Y h_*[k] = [i^Y \circ h \circ k]$. But we have $g(i^X(k(s))) = g(k(s)) = f(k(s))$, using $k(s) \in X$, and $i^Y(h(k(s))) = h(k(s)) = f(k(s))$, using $k(s) \in A$. Thus $g \circ i^X \circ k = i^Y \circ h \circ k$, so the first square commutes.

The second square is similar. Let $[k] \in \pi_n(X, x_0)$. Then $f_* j_*^X [k] = [f \circ j^X \circ k]$ and $j_*^Y g_*[k] = [j^Y \circ g \circ k]$. We have $f(j^X(k(s))) = f(k(s))$ and $j^Y(g(k(s))) = g(k(s)) = f(k(s))$, so the second square commutes.

Let $[k] \in \pi_n(X, A, x_0)$. Then $h_* \partial^X [k] = [h \circ k|_\partial]$ and $\partial^Y f_*[k] = [(f \circ k)|_\partial]$, where $|_\partial$ means restriction to the appropriate boundary; for instance, if $k : (D^n, S^{n-1}, s_0) \to (X, A, x_0)$, then $k|_\partial : (S^{n-1}, s_0) \to (A, x_0)$. We have $h(k|_\partial(s)) = h(k(s)) = f(k(s))$, using definition of restriction and that $k(s) \in A$. We also have $(f \circ k)|_\partial(s) = f(k(s))$ by definition. Thus, the third square commutes.

Since each of the three squares shown above commute, and since they represent all possible squares in the diagram between the long exact sequences, the long exact sequence is natural. $\square$

Problem 3

(1) Let $n \geq 2$. Suppose a sum $f +' g$ of maps $f, g : (I^n, \partial I^n) \to (X, x_0)$ is defined using a coordinate of I^n other than the first coordinate as in the usual sum $f + g$. Verify the formula $(f + g) +' (h + k) = (f +' h) + (g +' k)$ and deduce that $f +' k \simeq f + k$ so the two sums agree on $\pi_n(X, x_0)$, and also that $g +' h \simeq h + g$, so the addition is abelian.

(2) Show that if $\varphi : X \to Y$ is a homotopy equivalence, then the induced homomorphisms $\varphi_* : \pi_n(X, x_0) \to \pi_n(Y, \phi(x_0))$ are isomorphisms for all n.

Proof. (1) For the first assertion, we expand and simplify definitions:

$$\begin{aligned}
((f+g) +' (h+k))(s_1, \ldots, s_n) &= \begin{cases} (f+g)(s_1, \ldots, 2s_i, \ldots, s_n) & s_i \leq \frac{1}{2} \\ (h+k)(s_1, \ldots, 2s_i - 1, \ldots, s_n) & s_i \geq \frac{1}{2} \end{cases} \\
&= \begin{cases} f(2s_1, \ldots, 2s_i, \ldots, s_n) & s_1, s_i \leq \frac{1}{2} \\ g(2s_1 - 1, \ldots, 2s_i, \ldots, s_n) & s_1 \geq \frac{1}{2} \geq s_i \\ h(2s_1, \ldots, 2s_i - 1, \ldots, s_n) & s_1 \leq \frac{1}{2} \leq s_i \\ k(2s_1 - 1, \ldots, 2s_i - 1, \ldots, s_n) & s_1, s_i \geq \frac{1}{2} \end{cases} \\
&= \begin{cases} f(2s_1, \ldots, 2s_i, \ldots, s_n) & s_1, s_i \leq \frac{1}{2} \\ h(2s_1, \ldots, 2s_i - 1, \ldots, s_n) & s_1 \leq \frac{1}{2} \leq s_i \\ g(2s_1 - 1, \ldots, 2s_i, \ldots, s_n) & s_1 \geq \frac{1}{2} \geq s_i \\ k(2s_1 - 1, \ldots, 2s_i - 1, \ldots, s_n) & s_1, s_i \geq \frac{1}{2} \end{cases} \\
&= \begin{cases} (f +' h)(2s_1, \ldots, s_i, \ldots, s_n) & s_1 \leq \frac{1}{2} \\ (g +' k)(2s_1 - 1, \ldots, s_i, \ldots, s_n) & s_1 \geq \frac{1}{2} \end{cases} \\
&= ((f +' h) + (g +' k))(s_1, \ldots, s_n).
\end{aligned}$$

We make two observations. First, $f +' g$ is well-defined up to homotopy, i.e. if $f \simeq f'$ and $g \simeq g'$, then $f +' g \simeq f' +' g'$; the proof is no different than the proof for $f + g$ (up to changing some notation). Second, if $c : (I^n, \partial I^n) \to (X, x_0)$ is the constant map, then $f +' c \simeq f \simeq c +' f$ for any map $f : (I^n, \partial I^n) \to (X, x_0)$; once again, the proof is no different in this case.

We use these observations as follows. Starting from the equation $(f+g) +' (h+k) = (f +' h) + (g +' k)$, set g and h to be the constant map c. Since c is a homotopy identity for $+$ and $+'$, the left-hand side is homotopic to $f +' k$, and the right-hand side is homotopic to $f + k$. We have used that $+$ and $+'$ are well-defined up to homotopy to deduce, for instance, $(f + c) +' (c + k) \simeq f +' k$ from $f + c \simeq f$ and $c + k \simeq k$. Since we have one map which is homotopic to two others, those two are homotopic; $f +' k \simeq f + k$.

We can use the exact same reasoning to show that $+'$ is abelian up to homotopy, i.e. by replacing f and k by the constant map c (which I assume is Hatcher's intended solution). Alternatively, we have $f +' g \simeq f + g \simeq g + f \simeq g +' f$, using that $+$ and $+'$ agree up to homotopy and that $+$ is abelian up to homotopy.

(2) Let ψ be a homotopy inverse to φ. It is mentioned in Hatcher page 342 that homotopic maps induce the same homomorphisms on homotopy groups. In particular, $\psi \circ \varphi$ and $\varphi \circ \psi$ induce the same homomorphisms as the identity maps $\mathrm{id}_X : X \to X$ and $\mathrm{id}_Y : Y \to Y$, respectively. Furthermore, the induced homomorphism of a composition of maps is the composition of induced homomorphisms: $(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$ and $(\varphi \circ \psi)_* = \varphi_* \circ \psi_*$. Thus, to show φ_* is an isomorphism, it suffices to show that id_X (and by symmetry, id_Y) induces the identity automorphism. We only need to expand the definitions to see why this is true:

$$(\mathrm{id}_X)_*[f] = [\mathrm{id}_X \circ f] = [f],$$

since $\mathrm{id}_X \circ f = f$. $\square$

Problem 4

Show that the long exact homotopy sequence of the based pair (X, A, x_0) is exact at $\pi_n(A, x_0)$ for $n \geq 2$.

Proof. Consider the setup:

$$\pi_{n+1}(X, A, x_0) \xrightarrow{\partial} \pi_n(A, x_0) \xrightarrow{i_*} \pi_n(X, x_0).$$

Exactness of this sequence is equivalent to the following statement: Given $k : (S^n, s_0) \to (A, x_0)$, $i \circ k$ is nullhomotopic if and only if $k \simeq \tilde{k}|_{S^n}$ for some $\tilde{k} : (D^{n+1}, S^n, s_0) \to (X, A, x_0)$. In fact, an *a priori* stronger statement is true: $i \circ k$ is nullhomotopic if and only if $k = \tilde{k}|_{S^n}$ for some $\tilde{k} : (D^{n+1}, S^n, s_0) \to (X, A, x_0)$. In fact, A is irrelevant here; we will show that a map $k : (S^n, s_0) \to (X, x_0)$ is nullhomotopic if and only if it extends to a map $\tilde{k} : (D^{n+1}, s_0) \to (X, x_0)$. This result can then be applied to $i \circ k$ in the original notation.

Suppose k is nullhomotopic via a homotopy h, with $h(s, 1) = k(s)$. Since h is constant on $S^n \times \{0\}$, we get a well-defined map $\tilde{h} : S^n \times I/S^n \times \{0\} \to X$ induced by h. Note that $\tilde{h}(s_0, 1) = h(s_0, 1) = k(s_0) = x_0$, so $\tilde{h}$ is a based map $(S^n \times I/S^n \times \{0\}, (s_0, 1)) \to (X, x_0)$. Furthermore, $(S^n \times I/S^n \times \{0\}, (s_0, 1)) \cong (D^{n+1}, s_0)$, with $S^n \times \{1\}$ being identified with ∂D^{n+1}. Thus, $\tilde{h}$ determines a based map $\tilde{k} : (D^{n+1}, s_0) \to (X, x_0)$. For any $s \in \partial D^{n+1}$, we have $\tilde{k}(s) = \tilde{h}(s, 1) = h(s, 1) = k(s)$. Therefore, $\tilde{k}$ extends k.

Conversely, suppose k can be extended to $\tilde{k}$. Furthermore, assume that S^n and D^{n+1} are given as the unit sphere and unit ball in $\mathbb{R}^{n+1}$ respectively, with $s_0 = (0, \ldots, 0, 1)$. Define $h : S^n \times I \to X$ by $h(s, r) = \tilde{k}((1 - r)s_0 + rs)$. This is continuous since it is a composition of continuous maps, with the inner map being continuous by standard arguments for the topological vector space structure on $\mathbb{R}^{n+1}$. We have $h(s, 0) = \tilde{k}(s_0) = c(s)$, where $c : (S^n, s_0) \to (X, x_0)$ is the constant map. We also have $h(s, 1) = \tilde{k}(s) = k(s)$. Thus h is a homotopy between k and c, showing k is nullhomotopic as desired. $\square$