MATH 7590 Homework 1

Andrea Bourque

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1 Problem 1

Use the universal coefficient theorem (UCT) to determine the cohomology groups $H^n(S,\mathbb{Z})$ for S a closed surface. Also determine $H^n(S;\mathbb{Z}/2\mathbb{Z})$ for S a nonorientable closed surface.

Proof. First let S be orientable of genus $g \geq 0$. Then the $\mathbb{Z}$ homology is

$$H_n(S;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0, \\ \mathbb{Z}^{2g}, & n=1, \\ \mathbb{Z}, & n=2, \\ 0, & else. \end{cases}$$

The UCT says

$$0 \to \mathrm{Ext}(H_{n-1}(S;\mathbb{Z}),\mathbb{Z}) \to H^n(S;\mathbb{Z}) \to \mathrm{Hom}(H_n(S;\mathbb{Z}),\mathbb{Z}) \to 0$$

is exact. Since $\mathrm{Ext}(A,B)$ vanishes when A is free, and $H_n(S;\mathbb{Z})$ is always free, we get that $H^n(S;\mathbb{Z}) \cong \mathrm{Hom}(H_n(S;\mathbb{Z}),\mathbb{Z})$. A morphism from a free (finitely generated) abelian group to $\mathbb{Z}$ is determined by choice of integer for each generator, so $\mathrm{Hom}(\mathbb{Z}^k,\mathbb{Z}) \cong \mathbb{Z}^k$. Thus,

$$H^n(S;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0, \\ \mathbb{Z}^{2g}, & n=1, \\ \mathbb{Z}, & n=2, \\ 0, & else. \end{cases}$$

Now let S be non-orientable. Classification of surfaces says that S is the connected sum of $k > 0$ copies of $\mathbb{RP}^2$. The homology is

$$H_n(S;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n=0, \\ \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}, & n=1 \\ 0, & \text{else.} \end{cases}$$

Since $H_n(S;\mathbb{Z})$ is free for $n \neq 1$, we have $H^n(S;\mathbb{Z}) \cong \text{Hom}(H_n(S;\mathbb{Z}),\mathbb{Z})$ for $n \neq 2$. If $n \neq 0,1,2$, then $H_n(S;\mathbb{Z}) = 0$, so then $H^n(S;\mathbb{Z}) = 0$. We $H_0(S;\mathbb{Z}) = \mathbb{Z}$, so $H^0(S;\mathbb{Z}) = \mathbb{Z}$. Similarly, $H_1(S;\mathbb{Z}) = \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$, so $H^1(S;\mathbb{Z}) = \mathbb{Z}^{k-1}$ (the torsion part contributes nothing to the Hom).

To compute $H^2(S;\mathbb{Z})$, we must calculate $\text{Ext}(H_1(S;\mathbb{Z}),\mathbb{Z})$. The functor $\text{Ext}(-,B)$ commutes with direct sums, so we have $\text{Ext}(H_1(S;\mathbb{Z}),\mathbb{Z}) = \text{Ext}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})$, since the free part vanishes. Another property of Ext is that $\text{Ext}(\mathbb{Z}/m\mathbb{Z},B) = B/mB$. Thus, $\text{Ext}(H_1(S;\mathbb{Z}),\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Then, the UCT gives the exact sequence

$$0 \to \mathbb{Z}/2\mathbb{Z} \to H^2(S;\mathbb{Z}) \to \text{Hom}(H_2(S;\mathbb{Z}),\mathbb{Z}) \to 0.$$

Since $H_2(S;\mathbb{Z}) = 0$, we get $H^2(S;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. In summary,

$$H^n(S;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0, \\ \mathbb{Z}^{k-1}, & n = 1, \\ \mathbb{Z}/2\mathbb{Z}, & n = 2, \\ 0, & \text{else.} \end{cases}$$

Now, we compute the $\mathbb{Z}/2\mathbb{Z}$ cohomology of S. Again using the fact that $H_n(S;\mathbb{Z})$ is free for $n \neq 1$, we have $H^n(S;\mathbb{Z}/2\mathbb{Z}) \cong \text{Hom}(H_n(S;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$ for $n \neq 2$. For $n \neq 0,1,2$, we have $H_n(S;\mathbb{Z}) = 0$, so $H^n(S;\mathbb{Z}/2\mathbb{Z}) = 0$. We have $H_0(S;\mathbb{Z}) = \mathbb{Z}$, and $\text{Hom}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$, so $H^0(\mathbb{Z};\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. We also have $H_1(S;\mathbb{Z}) = \mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z}$, and $\text{Hom}(\mathbb{Z}^{k-1} \oplus \mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = \text{Hom}(\mathbb{Z},\mathbb{Z}/2\mathbb{Z})^{k-1} \oplus \text{Hom}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z}) = (\mathbb{Z}/2\mathbb{Z})^k$.

Next, we must compute $\text{Ext}(H_1(S;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$. We can use the same observations as before to reduce down to $\text{Ext}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z})$, and this is $(\mathbb{Z}/2\mathbb{Z})/2(\mathbb{Z}/2\mathbb{Z})$. But $2(\mathbb{Z}/2\mathbb{Z}) = 0$, so we just have $\text{Ext}(H_1(S;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. Once again, $H_2(S;\mathbb{Z}) = 0$ and UCT gives $H^2(S;\mathbb{Z}/2\mathbb{Z}) \cong \text{Ext}(H_1(S;\mathbb{Z}),\mathbb{Z}/2\mathbb{Z})$, so $H^2(S;\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}$. In summary,

$$H^n(S;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & n = 0,2 \\ (\mathbb{Z}/2\mathbb{Z})^k, & n = 1, \\ 0, & \text{else.} \end{cases}$$

$\square$

2 Problem 2

For the real projective space $\mathbb{RP}^k$, recall that

$$H_n(\mathbb{RP}^k) = \begin{cases} \mathbb{Z}, & n = 0, \\ \mathbb{Z}/2\mathbb{Z}, & n \leq k-1 \text{ and } n \text{ is odd}, \\ \mathbb{Z}, & n = k \text{ is odd}, \\ 0, & \text{else}. \end{cases}$$

Determine $H^n(\mathbb{RP}^k;\mathbb{Z})$ and $H^n(\mathbb{RP}^k;\mathbb{Z}/2\mathbb{Z})$ for all k and all n.

Proof. Since $H_n(\mathbb{RP}^k;\mathbb{Z})$ is free for even n, it follows that the Ext term in the UCT exact sequence vanishes for odd n. In particular, $H^n(\mathbb{RP}^k;G) = \mathrm{Hom}(H_n(\mathbb{RP}^k;\mathbb{Z}),G)$, where n is odd and G is the coefficient group. In the case $G=\mathbb{Z}$, using $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z})=0$, we get that the only non-zero cohomology for odd n is $H^k(\mathbb{RP}^k;\mathbb{Z})=\mathbb{Z}$, if k is odd. In the case $G=\mathbb{Z}/2\mathbb{Z}$, using $\mathrm{Hom}(\mathbb{Z}/2\mathbb{Z},\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$, we get $H^n(\mathbb{RP}^k;\mathbb{Z}/2\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$ for odd $n \leq k$, and 0 for odd $n > k$.

So far, we have computed H^n for both coefficient groups and all odd n. Next, consider $n=0$. Again, the Ext term vanishes, and $H_0(\mathbb{RP}^k;\mathbb{Z})=\mathbb{Z}$, so $H^0(\mathbb{RP}^k;G)=\mathrm{Hom}(\mathbb{Z},G)=G$. Next, consider n even and $n>k$. Then $n-1>k-1$, so $H_{n-1}(\mathbb{RP}^k;\mathbb{Z})$ is free, and $H_n(\mathbb{RP}^k;\mathbb{Z})=0$, so $H^n(\mathbb{RP}^k;G)=\mathrm{Hom}(0,G)=0$. Finally, consider n even and $0<n\leq k$. Then $H_n(\mathbb{RP}^k;\mathbb{Z})=0$, so the Hom term in the UCT exact sequence vanishes, giving $H^n(\mathbb{RP}^k;G)=\mathrm{Ext}(H_{n-1}(\mathbb{RP}^k;\mathbb{Z}),G)$. Since $n\leq k$ and n is even, we have $n-1\leq k-1$ and $n-1$ is odd, so $H_{n-1}(\mathbb{RP}^k;\mathbb{Z})=\mathbb{Z}/2\mathbb{Z}$. Then $H^n(\mathbb{RP}^k;G)=\mathrm{Ext}(\mathbb{Z}/2\mathbb{Z},G)=G/2G$. For both $G=\mathbb{Z}$ and $G=\mathbb{Z}/2\mathbb{Z}$, we have $G/2G=\mathbb{Z}/2\mathbb{Z}$, so $H^n(\mathbb{RP}^k;G)=\mathbb{Z}/2\mathbb{Z}$ in this case.

To summarize:

$$H^n(\mathbb{RP}^k;\mathbb{Z}) = \begin{cases} \mathbb{Z}, & n = 0 \text{ or } n = k \text{ if } k \text{ is odd}, \\ \mathbb{Z}/2\mathbb{Z}, & 0 < n \leq k \text{ and } n \text{ is even}, \\ 0, & \text{else}, \end{cases}$$

$$H^n(\mathbb{RP}^k;\mathbb{Z}/2\mathbb{Z}) = \begin{cases} \mathbb{Z}/2\mathbb{Z}, & 0 \leq n \leq k \\ 0, & \text{else}. \end{cases}$$

$\square$

3 Problem 3

Use the cellular map $\mathbb{RP}^2 \to S^2$ which collapses the 1-cell to a point to show that the splitting in the cohomology Universal Coefficient Theorem is not natural with respect to homomorphisms induced by continuous maps.

Proof. We take our coefficients (in homology and cohomology) to be $\mathbb{Z}$, as a $\mathbb{Z}$-module. We take the standard decomposition of $\mathbb{RP}^2$ into three cells, one in each of the dimensions 0,1,2. We also take the standard decomposition of S^2 into two cells, one in dimension 0 and one in dimension 2. The map $\mathbb{RP}^2 \to S^2$ sends the 0- and 1-cell to the 0-cell, and the 2-cell to the 2-cell. This map induces the following maps at the level of chains:

$$\begin{tikzcd}
\text{dim:} & 2 & 1 & 0 \\
\Delta_*(\mathbb{RP}^2): & \mathbb{Z} \arrow[r, "\times 2"] \arrow[d, "\mathrm{id}"'] & \mathbb{Z} \arrow[r, "0"] \arrow[d] & \mathbb{Z} \arrow[d, "\mathrm{id}"] \\
\Delta_*(S^2): & \mathbb{Z} \arrow[r] & 0 \arrow[r] & \mathbb{Z}
\end{tikzcd}$$

The induced maps in homology are:

$$\begin{tikzcd}
\text{dim:} & 2 & 1 & 0 \\
H_*(\mathbb{RP}^2): & 0 \arrow[d] & \mathbb{Z}/2\mathbb{Z} \arrow[d] & \mathbb{Z} \arrow[d] \\
H_*(S^2): & \mathbb{Z} & 0 & \mathbb{Z}
\end{tikzcd}$$

Taking duals of the maps at the level of chains, we get the maps at the level of cochains:

$$\begin{tikzcd}
\text{dim:} & & & \\
\Delta^*_{\mathbb{Z}}(\mathbb{RP}^2): & \mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\times 2"] & \mathbb{Z} \\
\Delta^*_{\mathbb{Z}}(S^2): & \mathbb{Z} \arrow[r] \arrow[u, "\mathrm{id}"] & 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[u, "\mathrm{id}"']
\end{tikzcd}$$

Then the induced maps in cohomology are:

$$\begin{tikzcd}
\text{dim:} & 0 & 1 & 2 \\
H^*(\mathbb{RP}^2;\mathbb{Z}) : & \mathbb{Z} & 0 & \mathbb{Z}/2\mathbb{Z} \\
H^*(S^2;\mathbb{Z}) : & \mathbb{Z} \arrow[u, "\mathrm{id}"] & 0 \arrow[u] & \mathbb{Z} \arrow[u, "\mathrm{mod}\ 2"]
\end{tikzcd}$$

We take the sequences given by UCT for $n = 2$:

$$\begin{tikzcd}
 & & \mathrm{Ext} & H^2 & \mathrm{Hom} & \\
\mathbb{RP}^2 : & 0 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r, "g_1"] & 0 \arrow[r] & 0 \\
S^2 : & 0 \arrow[r] & 0 \arrow[r] \arrow[u] & \mathbb{Z} \arrow[r, "g_2"'] \arrow[u, "\mathrm{mod}\ 2"] & \mathbb{Z} \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

The map g_1 must be the 0 map, and the map g_2 is an isomorphism, since the bottom row is short exact with first term 0. Splitting of a short exact sequence is equivalent to existence of left inverse of the map from the middle term. We now show that any choices of left inverses s_i for the maps g_i, the following diagram does not commute:

$$\begin{tikzcd}
\mathbb{Z}/2\mathbb{Z} & 0 \arrow[l, "s_1"'] \\
\mathbb{Z} \arrow[u, "\mathrm{mod}\ 2"] & \mathbb{Z} \arrow[l, "s_2"] \arrow[u]
\end{tikzcd}$$

No matter what s_1 is (although there is only one possibility), the composition along the right and top side is the 0 map. Since g_2 is an isomorphism, s_2 must be its inverse g_2^{-1}. Furthermore, g_2 is either the identity or negative identity map, both of which are their own inverses, and both of which induce bijections between even and odd elements of $\mathbb{Z}$. Therefore, the composition along the bottom and left side is the mod 2 map. Thus, the diagram does not commute, which means the splitting is not natural. $\square$

4 Problem 4

Use pinch maps to determine the ring structure of $H^*(\Sigma_g; \mathbb{Z})$, where Σ_g is a closed surface of genus $g \geq 2$.

Proof. Consider the plane model of Σ_g: it is a $4g$-gon with certain pairs of edges identified, and all the vertices identified. In particular, pick some vertex of the polygon, call it v_1, and proceed counter-clockwise to the next vertex, call it v_2. Label the path v_1 to v_2 by x_{11}. Then follow v_2 to v_3, and label it x_{12}. The path from v_4 to v_3 (this goes clockwise) is also labelled by x_{11}, and the path from v_5 to v_4 is also labelled by x_{12}. The repeated labels means that these edges are identified to form the surface. We have labeled 4 of our $4g$ edges; the edges in between v_{4i-3} and v_{4i+1} are labelled $x_{i1}, x_{i2}, x_{i1}, x_{i2}$ similarly, with the first two oriented counter-clockwise and the last two oriented clockwise. Under the identification of the edges with the same labels, all of the vertices v_i are identified to one vertex, call it v. Call the interior of the polygon F. This gives a CW cell decomposition of Σ_g.

Now we consider the cellular boundary maps of this CW structure. Since Σ_g is connected, we must have $H_0(\Sigma_g) = \mathbb{Z}$. Since we have a single 0-cell, the boundary map from 1-chains to 0-chains must be the 0 map. As for the map from 2-chains to 1-chains, we do need to compute the degrees of certain maps $\chi_{ij} : S^1 \to S^1$ where $i \in \{1, \ldots g\}$ and $j \in \{1, 2\}$ defined as follows. First, identify S^1 with the topological boundary of F (here I am abusing notation and referring to F as both the interior of the polygon and the disk which is attached to the 1-skeleton). Then, we take the attaching map $S^1 \to \Sigma_g^1$, where Σ_g^1 is the 1-skeleton of our cell decomposition. In particular, Σ_g^1 is a wedge sum of $2g$ circles, each labeled by x_{kl} for $k \in \{1, \ldots g\}$ and $l \in \{1, 2\}$. For the map χ_{ij}, the next step collapses all of the circles except for x_{ij}. Since this is a circle, we have ended with our codomain. The resulting composition can be described (in our case) in words as follows: trace the boundary of the 2-cell along one of the circles. By our polygon construction, each edge x_{ij} shows up exactly twice with opposite orientations. This means the resulting map $S^1 \to S^1$ goes around once in one orientation, and then again in the other orientation. This is a degree 0 map. Thus, the cellular boundary map from 2-chains to 1-chains is the 0 map. In particular, we can identify the generators of $H_i(\Sigma_g)$ with the i-cells. Furthermore, since the dual of a 0 map is a 0 map, we can also identify the generators of $H^i(\Sigma_g; \mathbb{Z})$ with the duals to the i-cells (which I will denote by putting a hat on the corresponding cell).

Let us now discuss a corresponding cell structure on the wedge sum $\bigvee^g \Sigma_1$. Here we take a single 0-cell w and $2g$ 1-cells y_{ij} as in Σ_g; however, we have g 2-cells, corresponding to the g tori; call them $F_1, \ldots F_g$. Explicitly, F_i is attached to the two circles y_{i1}, y_{i2} by going around y_{i1} in one orientation, y_{i2} in one orientation, y_{i1} in the opposite orientation, and y_{i2} in the opposite orientation. Just as in the case of Σ_g, the cellular boundary maps are 0. Furthermore, the homology and cohomology can be identified with chains and cochains, just as for Σ_g.

Now, let us describe the "pinch" map. Consider the description of Σ_g as a $4g$-gon. Consider the subset of Σ_g formed by the cycle of vertices $v_1, v_5, ..., v_{4g-3}, v_1$ and the "interior" (I'm not sure of a formal way to describe this that doesn't depend on the drawing of the $4g$-gon) of this cycle. The map $p : \Sigma_g \to \bigvee^g \Sigma_1$ is defined by quotienting out this region. In terms of chains (or homology classes), we say $p_*(F) = F_1 + \cdots + F_g$, $p_*(x_{ij}) = y_{ij}$, $p_*(v) = w$. Note that this is automatically a chain map, since all differentials are 0.

Now we dualize these maps. The dual of a 0 map is a 0 map, so all the differentials are 0. In terms of cochains and cohomology classes, we clearly have $p^*(\hat{y}_{ij}) = \hat{x}_{ij}$ and $p^*(\hat{w}) = \hat{v}$. Then we have $p^*(\hat{F}_i)(F) = \hat{F}_i(p_*(F)) = \hat{F}_i(\sum_j F_j) = \sum_j \delta_{ij} = 1$, so $p^*(\hat{F}_i) = \hat{F}$. In particular, p^* is an isomorphism on H^0 and H^1, and a surjection on H^2.

We want to compute the cup product structure on $H^*(\Sigma_g; \mathbb{Z})$. Of course, $\hat{v} \in H^0$ acts trivially, and $H^1 \smile H^2 = H^2 \smile H^1 = H^2 \smile H^2 = 0$ for dimension reasons. The only non-trivial computation is then $\hat{x}_{ij} \smile \hat{x}_{kl}$. But we have a cochain map isomorphism on H^1, so $\hat{x}_{ij} \smile \hat{x}_{kl} = p^*(\hat{y}_{ij} \smile \hat{y}_{kl})$. We already know the cup product structure on Σ_1 and on wedge sums of Σ_1; namely,

$$\hat{y}_{ij} \smile \hat{y}_{kl} = \begin{cases} 0 & i \neq k \text{ or } j = l, \\ \hat{F}_i & i = k \text{ and } j < l, \\ -\hat{F}_i & i = k \text{ and } j > l. \end{cases}$$

Thus,

$$\hat{x}_{ij} \smile \hat{x}_{kl} = \begin{cases} 0 & i \neq k \text{ or } j = l, \\ \hat{F} & i = k \text{ and } j < l, \\ -\hat{F} & i = k \text{ and } j > l. \end{cases}$$

$\square$

5 Problem 5

a. Show that the two definitions given in class of the Hopf invariant $\gamma(f)$ agree.
b. For $f : S^{2n-1} \to S^n$ and $g : S^n \to S^n$, show that $\gamma(gf) = (\deg(g))^2\gamma(f)$.
c. For $h : S^{2n-1} \to S^{2n-1}$ and $f : S^{2n-1} \to S^n$, show that $\gamma(fh) = \gamma(f)\deg(h)$.

Proof. a. No

b. Fix generators $[\alpha] \in H^{2n-1}(S^{2n-1};\mathbb{Z})$ and $[\beta] \in H^n(S^n;\mathbb{Z})$. Let $\beta \smile \beta = \delta c$, and let $f^*\beta = \delta d$. We know $[d \smile f^*\beta - f^*c] = \gamma(f)[\alpha]$. Note that c doesn't depend on the function $S^{2n-1} \to S^n$, but d does. In particular, when computing $\gamma(gf)$, we must pick d' such that $(gf)^*\beta = \delta d'$. But $(gf)^* = f^*g^*$ and $g^*[\beta] = \deg(g)[\beta]$, so we can take $d' = \deg(g)d$. Similarly, we note that $f^*(\beta \smile \beta) = \delta f^*c$, so

$$\delta(gf)^*c = (gf)^*(\beta \smile \beta) = f^*g^*\beta \smile f^*g^*\beta$$
$$\equiv f^*\deg(g)\beta \smile f^*\deg(g)\beta = (\deg(g))^2 f^*(\beta \smile \beta) = \delta(\deg(g))^2 f^*c.$$

Therefore, we can $(gf)^*c = (\deg(g))^2 f^*c$ in our computation of $\gamma(gf)$. Putting everything together, we have:

$$[d' \smile (gf)^*\beta - (gf)^*c] = \gamma(gf)[\alpha]$$
$$[\deg(g)d \smile \deg(g)f^*\beta - (\deg(g))^2 f^*c] = \gamma(gf)[\alpha]$$
$$(\deg(g))^2[d \smile f^*\beta - f^*c] = \gamma(gf)[\alpha]$$
$$(\deg(g))^2\gamma(f)[\alpha] = \gamma(gf)[\alpha]$$
$$(\deg(g))^2\gamma(f) = \gamma(gf).$$

c. Fix generators $[\alpha] \in H^{2n-1}(S^{2n-1};\mathbb{Z})$ and $[\beta] \in H^n(S^n;\mathbb{Z})$. Let $\beta \smile \beta = \delta c$, and let $f^*\beta = \delta d$. We know $[d \smile f^*\beta - f^*c] = \gamma(f)[\alpha]$. We know that $h^*[\alpha] = \deg(h)[\alpha]$ and $(fh)^* = h^*f^*$. To compute $\gamma(fh)$, we need d' such that $h^*f^*\beta = \delta d'$. We can pick $d' = h^*d$. Then

$$[d' \smile (fh)^*\beta - (fh)^*c] = \gamma(fh)[\alpha]$$
$$[h^*d \smile h^*f^*\beta - h^*f^*c] = \gamma(fh)[\alpha]$$
$$h^*[d \smile f^*\beta - f^*c] = \gamma(fh)[\alpha]$$
$$h^*\gamma(f)[\alpha] = \gamma(fh)[\alpha]$$
$$\gamma(f)\deg(h)[\alpha] = \gamma(fh)[\alpha]$$
$$\gamma(f)\deg(h) = \gamma(fh).$$

$\square$