MATH 7510 Homework 7

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1 Problem 7a

Compute the fundamental group G_n of the connected sum of n tori.

Proof. Imagine the connected sum as a $2n$-gon plane diagram (for convenience, see the next page). Each tori contributes two generates, say a_i, b_i for $i = 1, 2, ..., n$. Let A be an open disk in the middle of the plane diagram, and let $X - B$ be a closed disk contained in A. Thus $X = A \cup B$ and $A \cap B$ is an open annulus. Then we have $\pi_1(A) = 0$, the trivial group, since it is contractible. $\pi_1(B)$ is the free group on $2n$ generators, namely $a_1, ..., a_n, b_1, ..., b_n$. $\pi_1(A \cap B) = \mathbb{Z}$ since an annulus deformation retracts onto a circle.

The generator $1 \in \pi_1(A \cap B)$ is clearly mapped to the identity in $\pi_1(A)$, since $\pi_1(A)$ is trivial. In $\pi_1(B)$, the generator is mapped to the boundary of the $2n$-gon, which is $[a_1, b_1] \cdots [a_n, b_n]$. Finally, by Seifert van-Kampen, we know that the generators come from $\pi_1(A)$ and $\pi_1(B)$. Since A contributes no generators, there are $2n$ generators which all come from B. Since $\pi_1(B)$ has no relations on its generators, the only relations come from identifying where the generator of $\pi_1(A \cap B)$ is sent via the two different inclusions into $\pi_1(X)$. Thus, $G_n = \pi_1(X) = \langle a_1, ..., a_n, b_1, ..., b_n \mid [a_1, b_1] \cdots [a_n, b_n] \rangle$.

$\square$

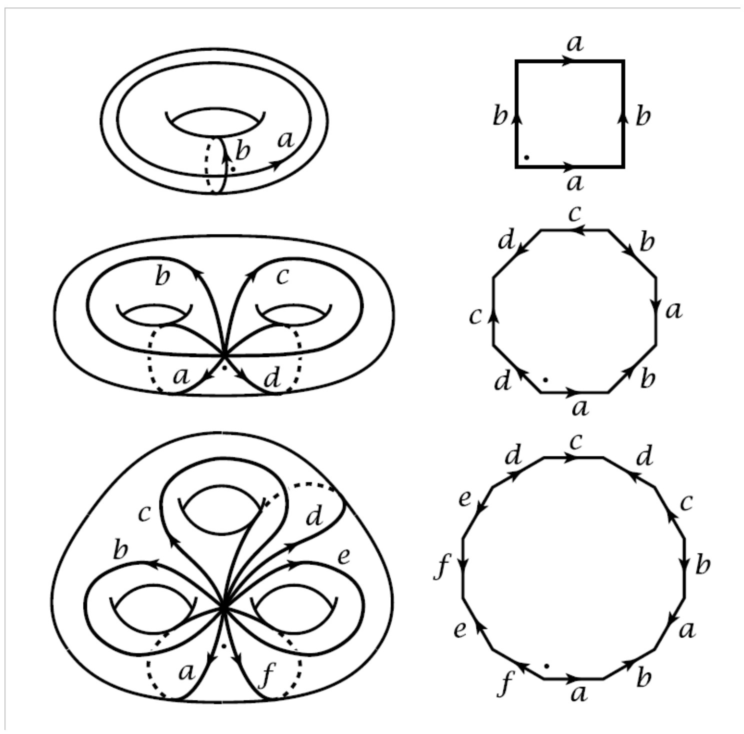

Figure 1: Hatcher page 5

2 Problem 7b

Show that for $n \geq 2$ this group G_n is not abelian by constructing a homomorphism from G_n onto the free group on two generators.

Proof. Let $F_2 = \langle a, b \mid \rangle$. Define a homomorphism $\varphi : G_n \to F_2$ via $\varphi(a_i) = \varphi(b_i) = a$ for $i = 1, ..., n-1$, and $\varphi(a_n) = \varphi(b_n) = b$. (It is a well known that it is sufficient to define a homomorphism on the generators of a group). Since homomorphisms preserve commutators, we have $\varphi([a_1, b_1] \cdots [a_n, b_n]) = [a, a] \cdots [b, b] = e$, so the relation in G_n becomes a trivial relation in the image. Thus, since $a, b \in \mathrm{im}\varphi$ and there are no relations, $\mathrm{im}\varphi = F_2$. This shows that G_n is not abelian, since we have $G_n / \ker \varphi \cong F_2$ is non-abelian, and the quotient of an abelian group must be abelian. $\square$

3 Problem 7c

Construct a homomorphism $G_n \twoheadrightarrow S_3$.

Proof. Write $S_3 = \langle a, b \mid a^2, b^2, (ab)^3 \rangle$. Then the previous homomorphism $\varphi : G_n \to S_3$ via $\varphi(a_i) = \varphi(b_i) = a$ for $i = 1, ..., n-1$, and $\varphi(a_n) = \varphi(b_n) = b$ also works, since there is no extra relation imposed. $\square$

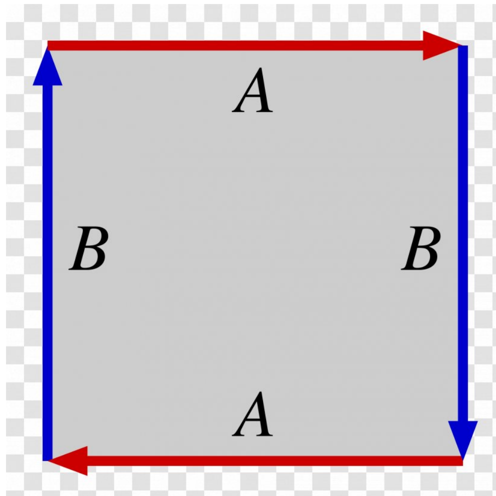

Figure 2: Klein bottle plane diagram

4 Problem 7d

Determine the fundamental group of the Klein bottle.

Proof. The Klein bottle can be written a plane diagram, so we can apply the same reasoning as in 7a. Indeed, A and B defined as in 7a, all that matters is where the generator of $\pi_1(A \cap B)$ is sent to under the inclusion into $\pi_1(B)$, which is the word given by the boundary, $aba^{-1}b$. Thus $\pi_1(K) = \langle a, b \mid aba^{-1}b \rangle$ by Seifert van Kampen. $\square$

5 Problem 7f

Compute the fundamental groups of $S^n \vee S^m$ for all m, n.

Proof. As in the case of $S^1 \vee S^1$, we take open sets A, B to be S^n, S^m with "a little extra", so that A, B are open and the "extra" is contractible, A is homeomorphic to S^n and B is homeomorphic to S^m. The intersection $A \cap B$ is then the two "extra" bits, so that $A \cap B$ is contractible. Since the intersection has trivial fundamental group, by Seifert van-Kampen we know that $\pi_1(S^n \vee S^m) = \pi_1(S^n) * \pi_1(S^m)$. For $n \geq 2$, $\pi_1(S^n) = 0$, so we have the following classification:

$$
\begin{aligned}
\pi_1(S^1 \vee S^1) &= \mathbb{Z} * \mathbb{Z} \\
\pi_1(S^1 \vee S^n) = \pi_1(S^n \vee S^1) &= \mathbb{Z} \mid n \geq 2 \\
\pi_1(S^n \vee S^m) &= 0 \mid n, m \geq 2
\end{aligned}
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