MATH 7510 Homework 6

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1 Problem 6a

Let A be a subspace of X with inclusion map $j : A \to X$ and a retraction $r : X \to A$. If $a_0 \in A$, show that the induced map r_* is surjective and that the induced map j_* is injective.

Proof. Let $[f]_A \in \pi_1(A, a_0)$. The loop f in A is also a loop in X, so $[f]_X$ is a well-defined element in $\pi_1(X, a_0)$. Furthermore, $r(f(a)) = f(a)$ for all $a \in A$. Thus $r_*([f]_X) = [r(f)]_A = [f]_A$, showing that r_* is surjective.

Let $[f]_A \in \pi_1(A, a_0)$ be such that $j_*([f]_A) = [e]_X$, the class of the identity loop in X. But $j_*([f]_A) = [j(f)]_X = [f]_X$, since f is also a loop in X. Thus $f \sim_X e$ in X. If φ is a path homotopy from f to e, then $r(\varphi)$ is also a path homotopy from f to e, since it preserves the loops f and e, which are contained in A. Furthermore, all values of $r(\varphi)$ are contained in A. Thus $f \sim_A e$ in A, implying $[f]_A = [e]_A$, so j_* is injective (a homomorphism with trivial kernel is injective). $\square$

2 Problem 6b

Show that a retract of a Hausdorff space is closed.

Proof. Let X be Hausdorff and A a retract with inclusion j and retraction r. Let $x \in X - A$. Then $r(x) \in A$, so $x \neq r(x)$. Then there are disjoint neighborhoods U, V of $x, r(x)$ respectively. Note that since $r(x) \in V$, $x \in r^{-1}(V)$, which is open by continuity. Then $W = U \cap r^{-1}(V)$ is a neighborhood of x. Suppose $a \in A \cap W$. Then $r(a) = a \in V$ and $a \in U$. This contradicts U, V being disjoint. Thus A and W are disjoint. W is a neighborhood of x contained in $X - A$, so A is closed. □

3 Problem 6c

Let A be a subspace of X with inclusion map j and $a_0 \in A$. Suppose A is a deformation retract of X with map $H : X \times I \to X$. Show that the induced map j_* is an isomorphism.

Proof. The map $r(x) = H(x, 1)$ gives a retraction $X \to A$, so by 6a we know j_* is injective. It suffices to show j_* is surjective. Let $[f]_X \in \pi_1(X, a_0)$. Then $F : I \times I \to X$ given by $F(t, s) = H(f(t), s)$ is a path homotopy from f to a loop $g(t) = F(t, 1)$ at a_0 in A. Indeed, F is a composition of continuous functions so it is continuous. It preserves the endpoints since $F(0, 1) = H(f(0), 1) = H(a_0, 1) = a_0$ and $g(1) = F(1, 1) = H(f(1), 1) = H(a_0, 1) = a_0$. It lies in A since $H(x, 1)$ lies in A for all x. Thus $f \sim_X g$, so $[f]_X = [g]_X$. Since g is a loop in A, $j_*([g]_A) = [j(g)]_X = [g]_X = [f]_X$, so j_* is surjective as desired. □

4 Problem 6d

Show that if A is a deformation retract of X and B is a deformation retract of A, then B is a deformation retract of X.

Proof. Let H be a deformation retraction from X to A, and let G be a deformation retraction from A to B. Let $F : X \times I \to X$ be $F(x, s) = G(H(x, s), s)$. F is continuous since it is the composition of continuous maps. $F(x, 0) = G(H(x, 0), 0) = G(x, 0) = x$. $F(x, 1) = G(H(x, 1), 1) \in B$ since $H(x, 1) \in A$ for all $x \in X$ and $G(x, 1) \in B$ for all $x \in A$. Let $b \in B$. Then $b \in A$. $F(b, s) = G(H(b, s), s) = G(b, s) = b$, since $H(b, s) = b$ by $b \in A$ and $G(b, s) = b$ by $b \in B$. Thus F is a deformation retraction of X to B. $\square$

5 Problem 6f

a) Show that I and $\mathbb{R}$ are contractible.

Proof. Let $i : I \to I$ be the identity map and $f : I \to I$ the constant map $f(x) = 0$. Let $F : I \times I \to I$ be $F(x,t) = (1-t)x$. F is continuous since it is a polynomial in its variables. It is also well defined (i.e. $F(x,t) \in I$) since $0 \le t \le 1$ implies $0 \le 1-t \le 1$, and this with $0 \le x \le 1$ implies $0 \le (1-t)x \le 1$. $F(x,0) = (1-0)x = x = i(x)$ and $F(x,1) = (1-1)x = 0 = f(x)$, so F is a homotopy between i and f. Thus i is nullhomotopic and I is contractible.

The same proof holds for $\mathbb{R}$: let $i : \mathbb{R} \to \mathbb{R}$ be the identity map and $f : \mathbb{R} \to \mathbb{R}$ be the constant map $f(x) = 0$. Then $F : \mathbb{R} \times I \to \mathbb{R}$ defined by $F(x,t) = (1-t)x$ is a homotopy between i and f, so $\mathbb{R}$ is contractible. □

b) Show that the retract of a contractible space is contractible.

Proof. Let X be contractible and let A be a retract of X with inclusion j and retraction r. Furthermore, let F be a homotopy between the identity map $i_X : X \to X$ and a constant map $f : X \to X$ given by $f(x) = x_0$. Let $a_0 = r(x_0)$. Let $i_A : A \to A$ be the identity map on A, and let $g : A \to A$ be the constant map $g(a) = a_0$. Let $G : A \times I \to A$ be $G(a,t) = r(F(a,t))$. G is continuous since it is a composition of continuous maps. $G(a,0) = r(F(a,0)) = r(i_X(a)) = r(a) = a$, since $F(x,0) = i_X(x)$ and $r(a) = a$ for $a \in A$. $G(a,1) = r(F(a,1)) = r(f(a)) = r(x_0) = a_0$, since $F(x,1) = f(x)$. Thus $G(a,0) = i_A(a)$ and $G(a,1) = g(a)$, so G is a homotopy between i_A and a constant map g. Thus A is contractible. □