MATH 7510 Homework 5

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1 Problem 1

Show that homotopy and path-homotopy are equivalence relations.

Proof. A map $f : X \to Y$ is homotopic to itself by the map $F : X \times [0,1] \to Y$ given by $F(x,t) = f(x)$. This is continuous since for open sets V in Y, $F^{-1}(V) = f^{-1}(V) \times [0,1]$ is open in $X \times [0,1]$.

If $f, g : X \to Y$ have $f \sim g$ via $F : X \times [0,1] \to Y$, then $g \sim f$ via $G : X \times [0,1] \to Y$ given by $G(x,t) = F(x, 1-t)$. Indeed, $G(x,0) = F(x,1) = g(x)$ and $G(x,1) = F(x,0) = f(x)$, and G is continuous since it is a composition of F with the continuous map $(x,t) \mapsto (x, 1-t)$ on $X \times [0,1]$.

If $f, g, h : X \to Y$ and $f \sim g$ via F and $g \sim h$ via G, then $f \sim h$ via $H(x,t) = F(x, 2t)$ for $t \in [0, \frac{1}{2}]$ and $H(x,t) = G(x, 2t-1)$ for $t \in [\frac{1}{2}, 1]$. H is continuous due to the pasting lemma, as $F(x,2t)$ and $G(x,2t-1)$ are continuous and $H(x, \frac{1}{2}) = F(x,1) = g(x) = G(x,0)$.

A path $f : [0,1] \to X$ has $f \sim_p f$ via $F(s,t) = f(s)$. Indeed, for all t, $f(0,t) = f(0)$ and $f(1,t) = f(1)$ so the endpoints are fixed.

If $f, g : [0,1] \to X$ are paths from x_0 to x_1 and $f \sim_p g$ via F, then $g \sim_p f$ via $G(s,t) = F(s, 1-t)$. For all t we have $G(0,t) = F(0, 1-t) = x_0$ and $G(1,t) = F(1, 1-t) = x_1$, and for all s we have $G(s,0) = F(s,1) = g(s)$ and $G(s,1) = F(s,0) = f(s)$.

If $f, g, h : [0,1] \to X$ are paths from x_0 to x_1 with $f \sim_p g$ via F and $g \sim_p h$ via G, then $f \sim_p h$ via $H(s,t) = F(s, 2t)$ for $t \in [0, \frac{1}{2}]$ and $H(s,t) = G(s, 2t-1)$ for $t \in [\frac{1}{2}, 1]$. Indeed, for all t, $H(0,t) = F(0,2t) = x_0$ or $H(0,t) = G(0, 2t-1) = x_0$ and $H(1,t) = F(1,2t) = x_1$ or $H(1,t) = G(1, 2t-1) = x_1$. For all s, $H(s,0) = F(s,0) = f(s)$ and $H(s,1) = G(s,1) = h(s)$. $\square$

2 Problem 2

Show that the product of two paths is associative up to path-homotopy.

Proof. Let f be a path from x_0 to x_1, let g be a path from x_1 to x_2, and let h be a path from x_2 to x_3. Let $p = f * (g * h), q = (f * g) * h$. Then we note that $q = p \circ \varphi$, where $\varphi(t) = 2t$ for $t \in [0, \frac{1}{4}]$, $\varphi(t) = t + \frac{1}{4}$ for $t \in [\frac{1}{4}, \frac{1}{2}]$, and $\varphi(t) = \frac{1}{2}t + \frac{1}{2}$ for $t \in [\frac{1}{2}, 1]$. This gives a path-homotopy between p, q since there is a homotopy between φ and the identity $t \mapsto t$: $\varphi_s(t) = (1-s)\varphi(t) + st$. Here $\varphi_0(t) = \varphi(t)$, $\varphi_1(t) = t$, so $q = p \circ \varphi_0(t), p = p \circ \varphi_1(t)$. □

3 Problem 3

Show that the product of a path $p(t)$ with its reverse path $p(1-t)$ is path-homotopic to the path that is constant $p(0)$.

Proof. The product $p(t) * p(1-t)$ is the path $q(t) = p(2t)$ for $t \in [0, \frac{1}{2}]$ and $q(t) = p(2-2t)$ for $t \in [\frac{1}{2}, 1]$. Let $F : [0,1] \times [0,1] \to X$ be $F(s,t) = p(2st)$ for $s \in [0, \frac{1}{2}]$ and $F(s,t) = p((2-2s)t)$ for $s \in [\frac{1}{2}, 1]$. For all t, $F(0,t) = p(0)$, $F(1,t) = p(2t-2t) = p(0)$. For all s, $F(s,0) = p(0)$, $F(s,1) = q(s)$. Thus F is the required path homotopy $p(0) \sim q(t)$. $\square$

4 Problem 4

Show that the product of a path $p(t)$ with the path that is constant $p(1)$ is path-homotopic to $p(t)$.

Proof. The product of $p(t)$ and $p(1)$ is $q(t) = p(2t)$ for $t \in [0, \frac{1}{2}]$ and $q(t) = p(1)$ for $t \in [\frac{1}{2}, 1]$. Let $F(s,t) = p((2-t)s)$ for $s \in [0, \frac{1}{2}(1+t)]$, $F(s,t) = p(1)$ for $s \in [\frac{1}{2}(1+t), 1]$. For all t, $F(0,t) = p(0)$, $F(1,t) = p(1)$. $F(s,0) = p(2s)$ for $s \in [0, \frac{1}{2}]$ and $F(s,0) = p(1)$ for $s \in [\frac{1}{2}, 1]$; in other words, for all s, $F(s,0) = q(s)$. For all s, $F(s,1) = p(s)$. Thus $q(t) \sim p(t)$ via F. $\square$