MATH 7510 Homework 4

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1 Problem 1

Show that for Tic-Tac-Toe on a torus, the first player has a winning strategy.

Proof. The first player should always play in the middle square.

If the second player plays on an edge, then the first player plays on a corner. The second player would then block the diagonal. The first player then plays another corner, which sets up two possible winning positions which cannot be simultaneously blocked by the second player.

O	O	X
	X	W
W		X

If the second player plays on a corner, then the first player should play an adjacent edge. The second player will then block the line. The first player then plays the other edge adjacent to the corner that the second player played, which sets up another line and a special torus diagonal, which cannot be simultaneously blocked.

	X	O
W	X	X
W	O	

□

2 Problem 2

Show that no two spaces among $[0,1], (0,1], (0,1)$ are homeomorphic.

Proof. Note that since $[0,1]$ is closed and bounded in $\mathbb{R}$, it is compact. The other two intervals are not compact in $\mathbb{R}$, since they are not closed. Thus $[0,1]$ is not homeomorphic to either $(0,1]$ or $(0,1)$.

For $(0,1]$, notice that one can remove the point 1, and the resulting space $(0,1)$ is connected. However, removing a point from $(0,1)$ results in a disconnected space $(0,a) \cup (a,1)$. Since $(0,a), (a,1)$ are open and $(0,a] \cap (a,1) = (0,a) \cap [a,1) = \emptyset$. Thus they are not homeomorphic. $\square$

3 Problem 3

Show that $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic for $n > 1$.

Proof. Removing a point from $\mathbb{R}$ results in a space which is disconnected, since it is the union of open intervals $(-\infty, a) \cup (a, \infty)$ and $(-\infty, a] \cap (a, \infty) = (-\infty, a) \cap [a, \infty) = \emptyset$. However, for $n > 1$, removing a point from $\mathbb{R}^n$ results in a connected space. To see this, take two points in $\mathbb{R}^n - \{a\}$. If the straight line connecting them does not pass through a, then that constitutes a path. If a does fall on the line, we can take a small deformation of the line which moves away from a, giving a path. Thus each $\mathbb{R}^n$ for $n > 1$ is (path) connected after removing a point, while $\mathbb{R}$ does not have this property, showing that they are not homeomorphic. $\square$

4 Problem 4

Show that the product of two path connected space is path connected. Show that the image of a path connected under a continuous map is path connected.

Proof. Let X, Y be path-connected. Let $(x_0, y_0), (x_1, y_1) \in X \times Y$. Let $x : [0,1] \to X, y : [0,1] \to Y$ be paths with $x(0) = x_0, x(1) = x_1, y(0) = y_0, y(1) = y_1$. Then consider $x \times y : [0,1] \times [0,1] \to X \times Y$ given by $x \times y(s,t) = (x(s), y(t))$, which is continuous by a previous homework result. Then the path $xy(t) = x \times y(t,t) = (x(t), y(t))$ is a path from (x_0, y_0) to (x_1, y_1), showing that $X \times Y$ is path-connected.

Let $f : X \to Y$ be surjective with X path-connected. Let $y_0, y_1 \in Y$, and let $x_0, x_1 \in X$ with $f(x_i) = y_i$ for $i = 0, 1$. Let $x : [0,1] \to X$ be a path with $x(0) = x_0, x(1) = x_1$. Then $y = f \circ x : [0,1] \to Y$ is a path from with $y(0) = f(x(0)) = y_0, y(1) = y_1$, since composition of continuous maps is continuous. Thus Y is path-connected. $\square$