MATH 7510 Homework 3

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1 Problem 1

Let $f : A \to B$ and $g : C \to D$ be continuous. Prove $f \times g$ is continuous.

Proof. Since the sets $U \times V$, where U, V are open in B, D respectively, form a basis for the topology on $B \times D$, it suffices to show that $(f \times g)^{-1}(U \times V)$ is open in $A \times C$. $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$, and since f, g are continuous, $f^{-1}(U)$ is open in A, and $g^{-1}(V)$ is open in C. Thus $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$ is open in $A \times C$. $\square$

2 Problem 2

Show that a retraction $r : X \to A$ is a quotient map.

Proof. Since $A \subset X$ and $r(a) = a$ for $a \in A$, we have r is surjective, since any $a \in A$ has at least one preimage, e.g. a. If U is open in A, then $r^{-1}(U)$ is open in X, since r is continuous. Now suppose $U \subset A$ and $r^{-1}(U)$ is open in X. Consider $V = r^{-1}(U) \cap A$. By definition, V is open in A. If $x \in V$, then since $x \in A$, $r(x) = x$. Since $x \in r^{-1}(U)$, $r(x) \in U$. Thus $x \in U$, showing $V \subset U$. Now let $x \in U$. Since $U \subset A$, $x \in A$. $x \in r^{-1}(U)$ since $r(x) = x \in U$. Thus $x \in V$. Thus $U = V$, showing that U is open in A. □

3 Problem 4

Let $f : [0,1] \to [0,1]$ be continuous. Prove there is some $x \in [0,1]$ such that $f(x) = x$.

Proof. Note that $g(x) = f(x) - x$ is also continuous on $[0,1]$. If $f(0) = 0$ or $f(1) = 1$, we are done. Thus, suppose $f(0) \neq 0$ and $f(1) \neq 1$, or $g(0) \neq 0$ and $g(1) \neq 0$. $g(0) = f(0) \in [0,1]$, but it is not zero, so we have $g(0) > 0$. $g(1) = f(1) - 1$, and $f(1) \in [0,1]$, so $g(1) \in [-1,0]$. Again, $g(1) \neq 0$, so $g(1) < 0$. Since $[0,1]$ is connected, the intermediate value theorem implies there is some $x \in (0,1)$ with $g(x) = 0$, which means $f(x) = x$. $\square$

4 Problem 5

Let $f : S^1 \to \mathbb{R}$ be continuous. Show that there is $x \in S^1$ such that $f(x) = f(-x)$.

Proof. Note that $g(x) = f(x) - f(-x)$ is also a continuous function $S^1 \to \mathbb{R}$. We have $g(-x) = f(-x) - f(x) = -g(x)$. If $g(x) \neq 0$, then $g(-x)$ has the opposite sign as $g(x)$. Then, since S^1 is connected, the intermediate value theorem implies that there must be some x' with $g(x') = 0$, or $f(x') = f(-x')$. □