MATH 7510 Homework 2

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1 Problem 1

Show that for $f : \mathbb{R} \to \mathbb{R}$, the $\varepsilon - \delta$ definition of continuity is equivalent to the topological definition.

Proof. ($\to$) Let f be continuous in the $\varepsilon - \delta$ sense. Let U be open in $\mathbb{R}$. Since $\emptyset$ is open, suppose $f^{-1}(U) \neq \emptyset$. Let $x \in f^{-1}(U)$, so that $f(x) \in U$. Since U is open, there is a basis element (a, b) such that $f(x) \in (a, b) \subseteq U$. Then $f(x) - a > 0$ and $b - f(x) > 0$. Let $0 < \varepsilon < \min\{f(x) - a, b - f(x)\}$. $0 < \varepsilon$ implies $f(x) - \varepsilon < f(x) < f(x) + \varepsilon$, $\varepsilon < f(x) - a$ implies $a < f(x) - \varepsilon$, and $\varepsilon < b - f(x)$ implies $f(x) + \varepsilon < b$. Thus $f(x) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subseteq (a, b) \subseteq U$. By continuity, there is $\delta > 0$ such that for any $y \in \mathbb{R}$ with $|y - x| < \delta$, $|f(y) - f(x)| < \varepsilon$. That is, if $y \in (x - \delta, x + \delta)$, then $f(y) \in (f(x) - \varepsilon, f(x) + \varepsilon) \subseteq U$. Thus $x \in (x - \delta, x + \delta) \subseteq f^{-1}(U)$, implying that $f^{-1}(U)$ is open.

($\leftarrow$) Let f be continuous in the topological sense. Let $x \in \mathbb{R}$ and $\varepsilon > 0$. $(f(x) - \varepsilon, f(x) + \varepsilon)$ is open, so $f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$ is open by continuity. Furthermore, $f(x) \in (f(x) - \varepsilon, f(x) + \varepsilon)$, so $x \in f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$. Thus, there is a basis element (a, b) with $x \in (a, b) \subseteq f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$. Since $x \in (a, b)$, $x - a > 0$ and $b - x > 0$. Let $0 < \delta < \min\{x - a, b - x\}$. $\delta > 0$ implies $x - \delta < x < x + \delta$, $\delta < x - a$ implies $x - \delta > a$, and $\delta < b - x$ implies $x + \delta < b$. Thus $x \in (x - \delta, x + \delta) \subseteq (a, b) \subseteq f^{-1}((f(x) - \varepsilon, f(x) + \varepsilon))$. In other words, if $y \in \mathbb{R}$ such that $|y - x| < \delta$, then $|f(y) - f(x)| < \varepsilon$. $\square$

2 Problem 2

Show that if A is closed in Y and Y is closed in X, then A is closed in X.

Proof. We know $A = C \cap Y$ for some closed set C in X. Since Y is also closed in X, and the intersection of two closed sets is closed, A is closed in X. $\square$

3 Problem 3

Show that X is Hausdorff iff $\Delta = \{x \times x | x \in X\}$ is closed in $X \times X$.

Proof. ($\rightarrow$) Suppose X is Hausdorff. Let $(p, q) \in X \times X - \Delta$. Since p, q are distinct points in X, there are disjoint open sets U, V such that $p \in U, q \in V$. If $(x, x) \in U \times V$, then $x \in U$ and $x \in V$, so $x \in U \cap V = \emptyset$, a contradiction. Thus $U \times V$ contains no points in Δ; it is an open set containing (p, q) and contained in $X \times X - \Delta$. Thus $X \times X - \Delta$ is open, so Δ is closed.

($\leftarrow$) Suppose Δ is closed in $X \times X$. Let $p, q \in X$ with $p \neq q$. Then $(p, q) \in X \times X - \Delta$. Since Δ is closed, $X \times X - \Delta$ is open, so there is a basis element $U \times V$, where U, V are open in X, such that $(p, q) \in U \times V \subseteq X \times X - \Delta$. If $U \cap V \neq \emptyset$, then for $x \in U \cap V$, $(x, x) \in U \times V$. But $(x, x) \in \Delta$. Thus $U \cap V = \emptyset$, and X is Hausdorff. □

4 Problem 4

Show that a subspace of a Hausdorff space is Hausdorff.

Proof. Let X be Hausdorff, and let Y be a subspace of X. Let $p, q \in Y$ with $p \neq q$. Since $p, q \in X$, there are disjoint open sets U, V in X such that $p \in U, q \in V$. Then let $U' = Y \cap U, V' = Y \cap V$. Since $p \in Y$ and $p \in U$, $p \in U'$; similarly, $q \in V'$. By definition of subspace topology, U', V' are open in Y. $U' \cap V' = (U \cap V) \cap Y = \emptyset \cap Y = \emptyset$, so U', V' are disjoint open sets in Y which contain p, q respectively. Thus Y is Hausdorff. $\square$