MATH 7330 Homework 4

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Problem VI.4

(a) Let X and Y be Banach spaces. Prove that a theorem for $\mathcal{L}(X,Y)$ analogous to Theorem VI.1 holds if Y is weakly sequentially complete (every weak Cauchy sequence has a weak limit).

(b) Prove that if a Banach space is reflexive, then it is weakly sequentially complete.

Proof. (a) We prove that if T_n is a sequence in $\mathcal{L}(X,Y)$ such that $\alpha(T_n x)$ converges for all $x \in X$ and all $\alpha \in Y^*$, then T_n weakly converges. Note that this reduces to Theorem VI.1 when $X = Y = \mathcal{H}$ is a Hilbert space. Fix $x \in X$. Since convergence implies Cauchy, we have $\alpha(T_n x)$ is Cauchy for all $\alpha \in Y^*$. In other words, $T_n x$ is weakly Cauchy. By hypothesis, $T_n x$ has a weak limit, which we denote by Tx. By definition of weak limit, this means we have $\alpha(T_n x)$ converges to $\alpha(Tx)$ for all $\alpha \in Y^*$. If we prove that the map $T : x \mapsto Tx$ is in $\mathcal{L}(X,Y)$, then the fact that $\alpha(T_n x) \to \alpha(Tx)$ for all x, α is exactly the condition that $T_n \xrightarrow{w} T$. Linearity of T follows immediately from linearity of limits and linearity of functionals. I wasn't able to figure out how to show T is bounded, but Prof. Han said it is fine if I don't finish this one.

(b) Let X be a reflexive Banach space, and let x_n be a weak Cauchy sequence. Thus, for each $\alpha \in X^*$, we have αx_n is Cauchy. Note that we can interpret this by using the functionals $\tilde{x}_n \in X^{**} = \mathcal{L}(X^*, \mathbb{C})$. We have, for each $\alpha \in X^*$, that $\tilde{x}_n \alpha$ is Cauchy, and furthermore converges since $\mathbb{C}$ is complete. Now, the elements of $\mathbb{C}^*$ are given by multiplication by a scalar, so we obviously have that $c(\tilde{x}_n \alpha)$ converges for all $\alpha \in X^*$ and all $c \in \mathbb{C}^*$. Furthermore, $\mathbb{C}$ is weakly sequentially complete, since weak Cauchy sequences in $\mathbb{C}$ are Cauchy and weak limits in $\mathbb{C}$ are limits; both statements are obtained by taking the identity functional. Thus, by part (a), $\tilde{x}_n \xrightarrow{w} \tilde{x}$ for some $\tilde{x} \in X^{**}$; it is important to note that the weak convergence here is to be interpreted as weak operator convergence. Since X is reflexive, we indeed have that $\tilde{x}$ is represented by some $x \in X$, i.e. $\tilde{x}\alpha = \alpha x$ for $\alpha \in X^*$. We now claim that $x_n \xrightarrow{w} x$. It suffices to show that $\alpha x_n \to \alpha x$ for all

$\alpha \in X^*$. Since $\tilde{x}_n \xrightarrow{w} \tilde{x}$, we have $\tilde{x}_n\alpha \to \tilde{x}\alpha$ for all $\alpha \in X^*$, which exactly means $\alpha x_n \to \alpha x$, as desired.

$\square$

Problem VI.7

Give an example to show that the range of a bounded operator need not be closed. Prove that if T is bounded, everywhere defined, and an isometry, then the range of T is closed.

Proof. Consider the map $T : \ell_\infty \to \ell_\infty$ given by $T((x_i)_i) = (x_i/i)_i$. T is linear because each map $x \mapsto x/i$ is linear. T is bounded because $|x_i/i| \leq |x_i|$, so $||T((x_i)_i)|| \leq ||(x_i)_i||$. Now, consider the sequence $((x_{i,j})_i)_j$ $= (1, \sqrt{2}, \ldots, \sqrt{j}, 0, \ldots)$ in ℓ^∞. Then $T((x_{i,j})_i) = (1, 1/\sqrt{2}, \ldots, 1/\sqrt{j}, 0, \ldots)$ converges to $y = (1, 1/\sqrt{2}, \ldots,)$. However, y is not in the range of T, since $(1, \sqrt{2}, \ldots)$ is not in ℓ^∞. Thus, the range of T is not closed.

Now suppose T is bounded, everywhere defined, and an isometry. Let x_n be a sequence such that $y_n = Tx_n$ converges to a point y. In particular, y_n is a Cauchy sequence. Since T is an isometry, we have $||x_n - x_m|| = ||T(x_n - x_m)||$ $= ||y_n - y_m||$, so x_n is also Cauchy. Thus x_n converges to some x. Since T is continuous, we have Tx_n converges to Tx. Thus $Tx = y$, so y is in the range of T. In other words, the range of T is closed. $\square$

Problem VI.8

(a) Let A be a self-adjoint bounded operator on a Hilbert space. Prove that the eigenvalues of A are real and that the eigenvectors corresponding to distinct eigenvalues are orthogonal.

(b) From the proof of Theorem VI.8 derive a universal (but λ-dependent) bound for the norm of the resolvent of a self-adjoint operator at a non-real $\lambda \in \mathbb{C}$.

Proof. (a) Let v be an eigenvector (I will always use eigenvector to mean non-zero) of A with eigenvalue λ. Then

$$\lambda(v, v) = (v, \lambda v) = (v, Av) = (Av, v) = (\lambda v, v) = \overline{\lambda}(v, v).$$

Since $(v, v) \neq 0$, we have $\lambda = \overline{\lambda}$, so $\lambda \in \mathbb{R}$. Now let v, w be two eigenvectors of A with distinct eigenvalues λ, μ. Then

$$\lambda(v, w) = (\lambda v, w) = (Av, w) = (v, Aw) = (v, \mu w) = \mu(v, w),$$

where in the first equality we use $\lambda \in \mathbb{R}$. Since $\lambda \neq \mu$, we must have $(v, w) = 0$ as desired.

(b) In the proof of Theorem VI.8, it is shown that $||(A-\lambda I)x||^2 \geq \mathrm{Im}(\lambda)^2||x||^2$. Since λ is non-real, $A-\lambda I$ is invertible by Theorem VI.8, so we get $||R_\lambda(A)x|| \leq \frac{1}{|\mathrm{Im}(\lambda)|}||x||$. In other words, the desired bound is $1/|\mathrm{Im}(\lambda)|$.

$\square$

Problem VI.9

(a) Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Prove that $||A|| = \sup_{||x||=1} |(Ax, x)|$.

(b) Find an example which shows that the conclusion of (a) need not be true if A is not self-adjoint.

Proof. (a) Let $S = \sup_{||x||=1} |(Ax, x)|$. First, note that $|(Ax, y)| \le ||Ax||||y|| \le ||A||||x||||y||$, where the first inequality is Cauchy-Schwarz. In particular, if $||x|| = 1$, then $|(Ax, x)| \le ||A||$. Thus $||A|| \ge S$, so it suffices to show $||A|| \le S$ Now, note that

$$(x + y, Ax + Ay) - (x - y, Ax - Ay) = 2(x, Ay) + 2(y, Ax)$$
$$= 2(x, Ay) + 2\overline{(Ax, y)} = 2(x, Ay) + 2\overline{(x, Ay)} = 4\mathrm{Re}((x, Ay)).$$

Next, we have

$$|(x, Ax)| = ||x||^2 \left| \left(\frac{x}{||x||}, A\frac{x}{||x||} \right) \right| \le ||x||^2 S.$$

While the first step assumes $x \ne 0$, the combined inequality $|(x, Ax)| \le ||x||^2 S$ is still true when $x = 0$, since both sides will be 0. Now, there is a constant c with $|c| = 1$ such that $(x, Acy) = |(x, Ay)|$; in particular, (x, Acy) is real and non-negative. Using this, and assuming $||x|| = ||y|| = 1$, we have

$$|(Ax, y)| = |(x, Ay)| = (x, Acy)$$
$$= \frac{1}{4}((x + cy, Ax + Acy) - (x - cy, Ax - Acy))$$
$$\le \frac{1}{4}(|(x + cy, Ax + Acy)| + |(x - cy, Ax - Acy)|)$$
$$\le \frac{1}{4}(||x + cy||^2 + ||x - cy||^2)S = \frac{1}{4}(2||x||^2 + 2||cy||^2)S = S,$$

where in the second-to-last step we use the parallelogram law. It follows that $||(Ax, -)|| \le S$. But $||Ax|| = ||(Ax, -)||$, so we get $||Ax|| \le S$ for all $||x|| = 1$; in other words, $||A|| \le S$ as desired.

(b) Consider $\mathbb{C}^2$ with its usual Hilbert space structure. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. A is not self-adjoint, since the adjoint in $\mathbb{C}^n$ is given by conjugate transpose, and the conjugate transpose of A is $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Now, clearly $||Ax||^2 = |x_2|^2 \le |x_1|^2 + |x_2|^2 = ||x||^2$, and we have equality for $x = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, so $||A|| = 1$. On the other hand, $|(Ax, x)| = |\overline{x}_2||x_1| = |x_1||x_2|$, and if $1 = ||x||^2 = |x_1|^2 + |x_2|^2$, then $|x_1||x_2| \leq \frac{1}{2}(|x_1|^2 + |x_2|^2) = \frac{1}{2}$ by the AM-GM inequality. In particular, $\sup\limits_{||x||=1} |(Ax, x)| \leq \frac{1}{2} \neq 1 = ||A||$.

$\square$

Problem VI.12

Let X be a Banach space and $T \in \mathcal{L}(X)$. Then show that,

(a) If λ is in the residual spectrum of T, then λ is in the point spectrum of T'.

(b) If λ is in the point spectrum of T, then λ is in either the point or the residual spectrum of T'.

Proof. (a) An eigenvector for T' with eigenvalue λ is a (non-zero) functional α such that $(T'\alpha)v := \alpha(Tv)$ is equal to $\lambda\alpha v$ for all $v \in X$. We can rewrite this as $\alpha((\lambda - T)v) = 0$ for all $v \in X$. In other words, α vanishes on the range of $\lambda - T$. Since α is continuous, it must also vanish on the closure of the range of $\lambda - T$. By hypothesis, the closure of the range of $\lambda - T$ is not X. By Hahn-Banach, we can find a non-zero α which vanishes on the closure of the range of $\lambda - T$, i.e. λ is in the point spectrum of T'.

(b) Let $Tv = \lambda v$ for $v \neq 0$. Suppose λ is not in the point spectrum of T'. If β is in the range of $\lambda - T'$, then there is a functional α such that $\beta = (\lambda - T')\alpha$. In particular, $\beta w = ((\lambda - T')\alpha)w = \alpha((\lambda - T)w)$ for $w \in X$. It follows that $\beta v = 0$ for all β in the range of $\lambda - T'$. It then follows that $\beta v = 0$ for all β in the closure of the range of $\lambda - T'$. Since $v \neq 0$, there is a functional α such that $\alpha v \neq 0$, by Hahn-Banach. Thus, the range of $\lambda - T'$ is not dense, so λ is in the residual spectrum of T'.

□