MATH 7330 Homework 3

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Problem 13

Use the uniform boundedness principle to provide an alternative proof of the Hellinger-Toeplitz theorem.

Proof. Let A be a self-adjoint operator on a Hilbert space $\mathcal{H}$. Consider the family $\mathcal{F}$ of bounded linear transformations $(Ax, \cdot)$ for $||x|| = 1$. Then for each $y \in \mathcal{H}$ and all $x \in \mathcal{H}$ such that $||x|| = 1$, we have $|(Ax, y)| = |(x, Ay)| \leq ||x||||Ay|| = ||Ay||$, where the inequality is Cauchy-Schwarz. Thus $\{|(Ax, y)| : ||x|| = 1\}$ is bounded for each y. Thus the family $\mathcal{F}$ satisfies the hypotheses of the uniform boundedness principle. We conclude that $\{||(Ax, \cdot)|| : ||x|| = 1\}$ is bounded. But $||(Ax, \cdot)|| = ||Ax||$, so $\{||Ax|| : ||x|| = 1\}$ is bounded. Thus, A is bounded. $\square$

Problem 15

Let $\mathcal{H}$ be a separable Hilbert space with an orthonormal basis $\{x_n\}$. Let $\{y_n\}$ be a sequence of elements of $\mathcal{H}$. Prove that the following two statements are equivalent:

(a) $(x, y_n) \to 0$ as $n \to \infty$ for all $x \in \mathcal{H}$.

(b) $(x_m, y_n) \to 0$ as $n \to \infty$ for each m, and $\{||y_n||\}$ is bounded.

Proof. (a) implies (b): Of course we have $(x_m, y_n) \to 0$ for each m. Now consider the family $\mathcal{F}$ of bounded linear transformations $(y_n, \cdot)$. For each $x \in \mathcal{H}$, we have $|(x, y_n)| = |(y_n, x)|$ is a sequence of real numbers that converges to 0, so $\{|(y_n, x)|\}$ is bounded. Thus, by the uniform boundedness principle, $\{||(y_n, \cdot)||\} = \{||y_n||\}$ is bounded.

(b) implies (a): Let $M > 0$ be such that $||y_n|| < M$ for all n. Let $x = \sum c_m x_m$. Then $||x||^2 = \sum |c_n|^2$. Let $\varepsilon > 0$, and let N be large so that $||x - z_N|| < \varepsilon/M$, where $z_N = \sum_{m \leq N} c_m x_m$. Then $(z_N, y_n) = \sum_{m \leq N} \overline{c_m}(x_m, y_n)$ by sesquilinearity and continuity of the inner product. Since this is a finite linear combination of terms that go to 0, we have $(z_N, y_n) \to 0$. Now $|(x, y_n) - (z_N, y_n)| = |(x - z_N, y_n)| \leq ||x - z_N|| ||y_n|| < \varepsilon$. Thus $|(x, y_n)\ | < \varepsilon + |(z_N, y_n)|$. Since $|(z_N, y_n)| \to 0$, and ε is arbitrary, we have $(x, y_n) \to 0$ as desired. □

Problem 16

A subset S of a Banach space X is called weakly bounded iff for all $\lambda \in X^*$, $\sup_{x \in S} |\lambda(x)| < \infty$. S is called strongly bounded iff $\sup_{x \in S} ||x|| < \infty$. Prove that these conditions are equivalent.

Proof. Suppose S is strongly bounded. Let $\lambda \in X^*$. Then $\sup_{x \in S} |\lambda(x)| \leq \sup_{x \in S} ||\lambda|| ||x|| = ||\lambda|| \sup_{x \in S} ||x|| < \infty$. Thus S is weakly bounded.

Suppose S is weakly bounded. Note that the statements $\sup_{x \in S} |\lambda(x)| < \infty$ and $\sup_{x \in S} ||x|| < \infty$ are the same as the statements $\{|\lambda(x)| : x \in S\}$ is bounded and $\{||x|| : x \in S\}$ is bounded. With this reframing in mind, we aim to apply the uniform boundedness principle. By Theorem III.4, there is an injective norm-preserving map $X \to X^{**}$ which sends x to $\tilde{x} : \lambda \mapsto \lambda(x)$. Let $\mathcal{F}$ be the family of bounded linear maps $\tilde{x}$ for $x \in S$. Then, by hypothesis, for each $\lambda \in X^*$, we have $\{|\tilde{x}(\lambda)| : \tilde{x} \in \mathcal{F}\}$ is bounded, since $|\tilde{x}(\lambda)| = |\lambda(x)|$. Therefore, $\mathcal{F}$ satisfies the hypothesis of the uniform boundedness principle, so $\{||\tilde{x}|| : \tilde{x} \in \mathcal{F}\}$ is bounded. Since the map $x \mapsto \tilde{x}$ is norm-preserving, we have $\{||x|| : x \in S\}$ is bounded, as desired. $\square$

Problem 18

Extend the Hellinger-Toeplitz theorem to include pairs of operators A, B satisfying $(Ax, y) = (x, By)$.

Proof. We show that A and B are bounded via the closed graph theorem. First, suppose $x_n \to x$ and $Bx_n \to y$. Then for any $z \in \mathcal{H}$, we have $(z, y) = (z, Bx_n)$ $= (Az, x_n) \to (Az, x) = (z, Bx)$, where we have used continuity of inner product. This implies $y - Bx$ is orthogonal to all elements in $\mathcal{H}$, so it is 0. Thus B is bounded. Now suppose $Ax_n \to y$. Then for any $z \in \mathcal{H}$, we have $(y, z) = (Ax_n, z)$ $= (x_n, Bz) \to (x, Bz) = (Ax, z)$, so $y = Ax$. Thus A is bounded. □

Problem 19

Let X be a Banach space in either of the norms $||\cdot||_1$ or $||\cdot||_2$. Suppose that $||\cdot||_1 \leq C||\cdot||_2$ for some C. Prove that there is a D with $||\cdot||_2 \leq D||\cdot||_1$.

Proof. Consider the identity map $(X, ||\cdot||_2) \to (X, ||\cdot||_1)$. By hypothesis, this map is bounded. It is obviously a linear bijection. By the inverse mapping theorem, the identity map $(X, ||\cdot||_1) \to (X, ||\cdot||_2)$ is bounded, which translates to $||\cdot||_2 \leq D||\cdot||_1$ for some D. □