MATH 7330 Homework 2

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Problem 1

(a) Prove that an inner product can be recovered from its norm by the polarization identity:

$$(x, y) = \frac{1}{4}(||x + y||^2 - ||x - y||^2 - i||x + iy||^2 + i||x - iy||^2).$$

(b) Prove that a normed linear space is an inner product space (i.e. that the form defined above is actually an inner product) if and only if the norm satisfies the parallelogram law.

Proof. (a) This comes down to expanding everything on the right hand side via $||z||^2 = (z, z)$ and sesquilinearity. For instance,

$$\begin{aligned} ||x + y||^2 &= (x, x) + (x, y) + (y, x) + (y, y), \\ -||x - y||^2 &= -(x, x) + (x, y) + (y, x) - (y, y), \\ -i||x + iy||^2 &= -i(x, x) + (x, y) - (y, x) - i(y, y), \\ i||x - iy||^2 &= i(x, x) + (x, y) - (y, x) + i(y, y). \end{aligned}$$

We can see that the terms involving (x, x), (y, x), (y, y) cancel out when we add up everything, while all of the (x, y) terms accumulate to give $4(x, y)$, as desired.

(b) First suppose that we have an inner product space. To verify the parallelogram, we can use the same strategy as above; expand the expressions $||x + y||^2$ and $||x - y||^2$ via sesquilinearity of the inner product, and then observe the desired cancellations. As above,

$$\begin{aligned} ||x + y||^2 &= (x, x) + (x, y) + (y, x) + (y, y), \\ ||x - y||^2 &= (x, x) - (x, y) - (y, x) + (y, y), \end{aligned}$$

so $||x + y||^2 + ||x - y||^2 = 2(x, x) + 2(y, y) = 2||x||^2 + 2||y||^2$.

Now suppose that we are given a normed linear space whose norm satisfies the parallelogram law. We will show that (x, y) as defined by the left hand side of the polarization identity satisfies each of the conditions to be an inner product.

(i) We want to show $(x, x) \geq 0$ with equality if and only if $x = 0$. We have

$$\begin{aligned}(x, x) &= \frac{1}{4}(||2x||^2 - ||0||^2 - i||(1+i)x||^2 + i||(1-i)x||^2) \\ &= \frac{1}{4}(4||x||^2 - 2i||x||^2 + 2i||x||^2) = ||x||^2.\end{aligned}$$

By definition of norm, $||x||^2 \geq 0$ with inequality if and only if $x = 0$, so we are done.

(ii) We want to show $(x, y+z) = (x, y)+(x, z)$. The strategy will be to add and subtract terms in the expression $(x, y+z)$ in order to get pairs of terms to which we can apply the parallelogram law; I have colored matching terms. We have

$$\begin{aligned}4(x, y+z) = ||x+y+z||^2 &+ ||x-y+z||^2 \\ -(||x-y-z||^2 &+ ||x-y+z||^2) \\ -i(||x+iy+iz||^2 &+ ||x-iy+iz||^2) \\ +i(||x-iy-iz||^2 &+ ||x-iy+iz||^2),\end{aligned}$$

and applying the parallelogram on each line gives

$$\begin{aligned}4(x, y+z) = 2(||x+z||^2 &+ ||y||^2) \\ -2(||x-y||^2 &+ ||z||^2) \\ -2i(||x+iz||^2 &+ ||iy||^2) \\ +2i(||x-iy||^2 &+ ||iz||^2).\end{aligned}$$

Since $(x, y+z)$ is symmetric in y and z, we can swap them on the right hand side to get

$$\begin{aligned}4(x, y+z) = 2(||x+y||^2 &+ ||z||^2) \\ -2(||x-z||^2 &+ ||y||^2) \\ -2i(||x+iy||^2 &+ ||iz||^2) \\ +2i(||x-iz||^2 &+ ||iy||^2).\end{aligned}$$

Now we will add the previous two equations together. The left hand side is just $8(x, y+z)$. On the right hand side, all of the terms in the right column (on the right side of each +) cancel each other out; I have colored matching terms. We get

$$\begin{aligned}8(x, y+z) = 2(&||x+y||^2 + ||x+z||^2)\\ -2(&||x-y||^2 + ||x-z||^2)\\ -2i(&||x+iy||^2 + ||x+iz||^2)\\ +2i(&||x-iy||^2 + ||x-iz||^2)\\ &= 8(x,y) + 8\,(x,z)\,.\end{aligned}$$

Thus $(x, y+z) = (x,y) + (x,z)$ as desired.

(iii) We want to show $(x, \alpha y) = \alpha(x,y)$ for scalars α. This is obviously true for $\alpha = 1$. Using the previous property with $y = z$ gives $(x, 2y) = 2(x,y)$. By induction, we have $(x, ny) = n(x,y)$ for positive integers n. For $n = 0$, we have

$$(x, 0y) = (x, 0) = \frac{1}{4}(||x||^2 - ||x||^2 - i||x||^2 + i||x||^2) = 0 = 0(x,y).$$

For $n = -1$, we have

$$\begin{aligned}(x, -y) &= \frac{1}{4}(||x-y||^2 - ||x+y||^2 - i||x-iy||^2 + i||x+iy||^2)\\ &= -\frac{1}{4}(||x+y||^2 - ||x-y||^2 - i||x+iy||^2 + i||x-iy||^2) = -(x,y).\end{aligned}$$

By combining this with the result for positive integers, we have $(x, ny) = n(x,y)$ for all integers n. For non-zero integer n, we have

$$(x, \frac{1}{n}y) = \frac{1}{n}\cdot n(x, \frac{1}{n}y) = \frac{1}{n}(x,y).$$

Thus $(x, \alpha y) = \alpha(x,y)$ for all $\alpha \in \mathbb{Q}$. Since the norm is continuous, (x,y) must also be continuous. Since real numbers are limits of rational numbers, we then have $(x, \alpha y) = \alpha(x,y)$ for all $\alpha \in \mathbb{R}$. Finally, we show that $(x, iy) = i(x,y)$:

$$\begin{aligned}(x, iy) &= \frac{1}{4}(||x+iy||^2 - ||x-iy||^2 - i||x-y||^2 + i||x+y||^2)\\ &= i\cdot\frac{1}{4}(||x+y||^2 - ||x-y||^2 - i||x+iy||^2 + i||x-iy||^2) = i(x,y).\end{aligned}$$

Thus, $(x, \alpha y) = \alpha(x,y)$ for all $\alpha \in \mathbb{C}$.

(iv) We want to show $(x,y) = \overline{(y,x)}$. This is simple: norms are real, so the only change upon conjugating (y,x) is changing the signs of the two terms with i in the front. We will also use $||-z||^2 = ||\pm iz||^2 = ||z||^2$, which follows since $|-1|^2 = |\pm i|^2 = 1$ and $||\alpha z|| = |\alpha|||z||$. We have

$$\begin{aligned}(y, x) &= \frac{1}{4}(||y+x||^2 - ||y-x||^2 - i||y+ix||^2 + i||y-ix||^2)\\ &= \frac{1}{4}(||x+y||^2 - ||x-y||^2 - i||x-iy||^2 + i||x+iy||^2).\end{aligned}$$

As discussed, conjugating this involves changing the signs of the two terms with i in the front, which evidently gives (x, y):

$$\overline{(y,x)} = \frac{1}{4}(||x+y||^2 - ||x-y||^2 + i||x-iy||^2 - i||x+iy||^2)$$
$$= \frac{1}{4}(||x+y||^2 - ||x-y||^2 - i||x+iy||^2 + i||x-iy||^2) = (x, y).$$

$\square$

Problem 2

Let $\mathcal{H}$ be a Hilbert space, $\mathcal{M}$ a closed subspace of $\mathcal{H}$, and suppose $x \in \mathcal{H}$. Let $y, z \in \mathcal{M}$ be two elements closest to x. Then $y = z$.

Proof. Let $d = ||x - y|| = ||x - z||$. By hypothesis, for any $w \in \mathcal{M}$, we have $d \leq ||x - w||^2$. In particular, this is true for $w = (y + z)/2$. By the parallelogram law,

$$||y - z||^2 = ||(x - y) - (x - z)||^2 = 2||x - y||^2 + 2||x - z||^2 - ||2x - (y + z)||^2,$$

and by the above inequality, we have

$$||y - z||^2 = 4d^2 - 4||x - (y + z)/2||^2 \leq 4d^2 - 4d^2 = 0.$$

Thus $y = z$. □

Problem 3

Let $\{S_\alpha\}_{\alpha\in A}$ be a linearly ordered collection of orthonormal sets in a Hilbert space. Prove that $S = \bigcup_{\alpha\in A} S_\alpha$ is an orthonormal set.

Proof. Let $x \in S$. Then $x \in S_\alpha$ for some $\alpha \in A$. Since S_α is an orthonormal set, we know $(x, x) = 1$. Thus $(x, x) = 1$ for all $x \in S_\alpha$. Now let y be another element of S, different from x. There is some $\beta \in A$ such that $y \in S_\beta$. Since $\{S_\alpha\}_{\alpha\in A}$ is linearly ordered, either $S_\alpha \subseteq S_\beta$ or $S_\beta \subseteq S_\alpha$. Either way, we find that x, y are elements of the same orthonormal set, meaning $(x, y) = 0$. Thus $(x, y) = 0$ for all distinct elements of S. Thus S is an orthonormal set. □

Problem 4

(a) Let V be an inner product space. Prove that the inner product can be extended to its completion $\tilde{V}$ as follows: First, show that if $x, y \in \tilde{V}$, $x_n, y_n \in V$, and $x_n \to x$, $y_n \to y$, then (x_n, y_n) converges. Then define $(x, y) = \lim_{n\to\infty}(x_n, y_n)$ and show that it is independent of the choice of convergent sequences. Finally, show that $(\cdot, \cdot)$ is an inner product on $\tilde{V}$.

(b) Prove the statement in (a) by applying the BLT theorem twice.

Proof. (a) Since x_n and y_n are convergent sequences, we have $||x_n - x_m||, ||y_n - y_m|| < \varepsilon$ for arbitrary $\varepsilon > 0$ and large enough n, m. Also, for large enough n, we can bound $||x_n||$ and $||y_n||$. For convenience, let $dx = x_n - x_m$ and $dy = y_n - y_m$. We have

$$\begin{aligned}|(x_n, y_n) - (x_m, y_m)| &\le |(dx, dy)| + |(x_m, dy)| + |(dx, y_m)| \\ &\le ||dx|| \cdot ||dy|| + ||x_m|| \cdot ||dy|| + ||dx|| \cdot ||y_m||,\end{aligned}$$

where the first inequality follows from linearity of inner product, and the second inequality is Cauchy-Schwarz. As previously mentioned, we can bound $||x_m||$ and $||y_m||$ and make $||dx||, ||dy||$ as small as we want. Thus (x_n, y_n) is Cauchy, and since $\tilde{V}$ is complete, it converges to some element which we denote by (x, y).

Of course, we should check that (x, y) depends only on x, y, not on the sequences x_n, y_n we chose. If we chose other sequences x'_n, y'_n that converge to x and y respectively, it suffices to show that $|(x_n, y_n) - (x'_n, y'_n)|$ converges to 0. Indeed, by the same method, we will get three terms involving norms of the sequence terms and norms of the differences $x_n - x'_n$, $y_n - y'_n$. The former terms are bounded and the latter terms can be made as small as we wish, since both terms on the right hand side $||x_n - x'_n|| \le ||x_n - x|| + ||x - x'_n||$ can be made as small as we want by definition. Thus (x_n, y_n) and (x'_n, y'_n) get arbitrarily close, meaning (x'_n, y'_n) must get arbitrarily close to (x, y).

Linearity in the second variable follows from linearity of limits. $\overline{(x, y)} = (y, x)$ follows from continuity of complex conjugation (we can think of it as a linear map $\mathbb{R}^2 \to \mathbb{R}^2$ with bound 1). $(x, x) \ge 0$ follows because $[0, \infty)$ is closed; $(x, x) = \lim(x_n, x_n)$ is the limit of things in $[0, \infty)$. Finally, suppose $(x, x) = 0$. This means for some sequence $x_n \in V$ converging to x, we have (x_n, x_n) converging to 0. By definition, this means that $||x_n|| = ||x_n - 0||$ converges to 0, so x_n converges to 0.

(b) Fix $x \in V$. Then $(x, \cdot)$ is a linear transformation $V \to \mathbb{C}$ by definition, and it is bounded by Cauchy-Schwarz. By the BLT theorem, there is a unique extension of $(x, \cdot)$ to a bounded linear transformation on $\tilde{V}$ with the same bound. Now for fixed $y \in \tilde{V}$, we have $\overline{(\cdot, y)}$ is a linear transformation

$V \to \mathbb{C}$. It is bounded because $|\overline{(x,y)}| \leq ||x|| \cdot ||y||$ by the previous usage of BLT. Thus we can apply BLT again to get $\overline{(x,y)}$ for $x, y \in \tilde{V}$. We take the conjugate of this to get $(\cdot,\cdot)$ on $\tilde{V}$. Now, the *proof* of BLT shows that in fact (x,y) for $x, y \in \tilde{V}$ is defined exactly as in part (a), so we have already done the hard work to show that this is indeed an inner product.

$\square$

Problem 5

Let V be an inner product space and let $\{x_n\}_{n=1}^N$ be an orthonormal set. Prove that

$$\left|\left|x - \sum_{n=1}^{N} c_n x_n\right|\right|$$

is minimized by choosing $c_n = (x_n, x)$.

Proof. Set $c_n = (x_n, x)$. Let $X = x - \sum c_n x_n$. By linearity, we see that $(x_n, X) = 0$ for all n. Let $\{a_n\}$ be any other set of N complex numbers. We need a slightly modified version of the Pythagorean theorem given in the book. Namely, if u, v are orthogonal, then $||u+v||^2 = ||u||^2 + ||v||^2$; this follows simply by expanding the definition of norm and using linearity. We then have

$$\left|\left|x - \sum_{n=1}^{N} a_n x_n\right|\right|^2 = \left|\left|X + \sum_{n=1}^{N}(c_n - a_n)x_n\right|\right|^2 = ||X||^2 + \left|\left|\sum_{n=1}^{N}(c_n - a_n)x_n\right|\right|^2$$
$$\geq ||X||^2.$$

Taking square roots gives the desired claim. □

Problem 6

Let $\mathcal{M}$ be any linear subset of a Hilbert space $\mathcal{H}$. Prove that $\mathcal{M}^\perp$ is a closed linear subspace and that $\overline{\mathcal{M}} = (\mathcal{M}^\perp)^\perp$.

Proof. Let (x_n) be a sequence of elements in $\mathcal{M}^\perp$, and suppose $x_n \to x \in \mathcal{H}$. Let $y \in \mathcal{M}$. If $y = 0$, then of course $(x, y) = 0$. Suppose $y \neq 0$, let $\varepsilon > 0$, and let N be an integer such that $n > N$ implies $||x - x_n|| < \varepsilon/||y||$. Then for $n > N$, $(x, y) = (x - x_n, y) + (x_n, y) = (x - x_n, y)$, and $|(x - x_n, y)| \leq ||x - x_n|| \cdot ||y|| < \varepsilon$, so $|(x, y)| < \varepsilon$ for all $\varepsilon > 0$. Thus $(x, y) = 0$. Since y is arbitrary, this shows $x \in \mathcal{M}^\perp$, i.e. $\mathcal{M}^\perp$ is closed.

That $M^\perp$ is linear follows from the properties of the inner product: If $x_1, \ldots, x_n \in \mathcal{M}^\perp$, $c_1, \ldots, c_n \in \mathbb{C}$, and $y \in \mathcal{M}$, then $(y, \sum c_n x_n) = \sum c_n (y, x_n) = \sum 0 = 0$.

By definition, $\mathcal{M} \subseteq (\mathcal{M}^\perp)^\perp$. By the above result, $(\mathcal{M}^\perp)^\perp$ is closed. Thus $\overline{\mathcal{M}} \subseteq (\mathcal{M}^\perp)^\perp$. In order to show the reverse inclusion, we first show that $\mathcal{M}^\perp = \overline{\mathcal{M}}^\perp$. Obviously, $\overline{\mathcal{M}}^\perp \subseteq \mathcal{M}^\perp$, since $\mathcal{M} \subseteq \overline{\mathcal{M}}$. On the other hand, $\mathcal{M}^\perp \subseteq \overline{\mathcal{M}}^\perp$ by continuity of inner products: If a sequence of elements $x_i \in \mathcal{M}$ converges to some $x \in \overline{\mathcal{M}}$ and $y \in \mathcal{M}^\perp$, then $(x, y) = (\lim x_i, y) = \lim (x_i, y) = \lim 0 = 0$. We then have two decompositions: $\mathcal{H} = \overline{\mathcal{M}} \oplus \mathcal{M}^\perp$ and $\mathcal{H} = (\mathcal{M}^\perp)^\perp \oplus \mathcal{M}^\perp$. Let $x \in (\mathcal{M}^\perp)^\perp$. By the first decomposition, $x = y + z$ for $y \in \overline{\mathcal{M}}$ and $z \in \mathcal{M}^\perp$. But we already showed that $\overline{\mathcal{M}} \subseteq (\mathcal{M}^\perp)^\perp$, so $y \in (\mathcal{M}^\perp)^\perp$. By the second decomposition, we must then have $z = 0$ and $x = y \in \overline{\mathcal{M}}$. This proves $(\mathcal{M}^\perp)^\perp \subseteq \overline{\mathcal{M}}$, as desired. □

Problem 7

Let $\mathcal{M}$ be a subspace of a Hilbert space $\mathcal{H}$. Let $f : \mathcal{M} \to \mathbb{C}$ be a linear functional on $\mathcal{M}$ with bound C. Prove that there is a unique extension of f to a continuous linear functional on $\mathcal{H}$ with the same bound.

Proof. Note that the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ in $\mathcal{H}$ is in fact the completion of $\mathcal{M}$. By the BLT theorem, we get a unique extension of f to a linear functional on $\overline{\mathcal{M}}$ with the same bound C. Next, we set f to be 0 on $\overline{\mathcal{M}}^{\perp}$. By projection theorem, this defines a linear functional on $\mathcal{H}$, and it clearly has the same bound.

To show this is unique, we only need to consider the possible extensions from $\overline{\mathcal{M}}$, thanks to BLT. If $\overline{\mathcal{M}} = \mathcal{H}$, we are done. Since $\overline{\mathcal{M}}$ is a Hilbert space, it has an orthonormal basis $\{x_\alpha\}_{\alpha \in A}$, and this extends to an orthonormal basis $\{x_\beta\}_{\beta \in B}$, where $A \subset B$. Since bounded linear transformations are linear and continuous, they are determined by their values on an orthonormal basis. So, we must show that $f(x_\beta) = 0$ for all $\beta \in B \setminus A$.

Let us take care of a special case first. Suppose f is the 0 functional on $\overline{\mathcal{M}}$. Then its operator norm (minimal bound C) on $\overline{\mathcal{M}}$ is 0. Conversely, any bounded linear functional with bound 0 is of course 0. Thus, the only way to extend f to a bounded linear functional on $\mathcal{H}$ with the same bound is to set $f = 0$ on $\mathcal{M}$. We henceforth suppose f is not the 0 functional on $\overline{\mathcal{M}}$.

Define $c_\beta = f(x_\beta)$ for all $\beta \in B$, and suppose this defines a bounded linear functional on $\mathcal{H}$ with the same bound C. Suppose $c_\beta \neq 0$ for some $\beta \in B \setminus A$. Let $x \in \overline{\mathcal{M}}$ be such that $f(x) \neq 0$. Suppose that the bound C is in fact taken to be the operator norm of f on $\overline{\mathcal{M}}$. Then an appropriate choice of x can make $C^2||x||^2 - |f(x)|^2$ as small as we want. At the same time, we can also make $|f(x)|$ as large as we want. Furthermore, by multiplying x by an appropriate scalar, we can assume $f(x) > 0$. Next, choose $a \in \mathbb{C}$ so that $f(ax_\beta) = ac_\beta > 0$. Hence $|f(x + ax_\beta)| = f(x) + ac_\beta$. Then the bound gives

$$f(x)^2 + 2ac_\beta f(x) + a^2 c_\beta^2 \leq C^2||x + ax_\beta||^2 = C^2||x||^2 + C^2|a|^2.$$

Note that the equality follows from the Pythagorean theorem; x and x_β are orthogonal. Then

$$f(x) \leq \left((C^2||x||^2 - f(x)^2) + C^2|a|^2 - a^2c_\beta^2\right)/(2ac_\beta).$$

As previously stated, $C^2||x||^2 - f(x)^2$ can be made as small as we want, so in fact we get $f(x) \leq (C^2|a|^2 - a^2c_\beta^2)/(2ac_\beta)$. But the two sides of the inequality are independent, and as mentioned before, we can make $f(x)$ as large as we want! Therefore, f so defined cannot have bound C, and this contradiction arose from the assumption $c_\beta \neq 0$. Thus we are done. □

Problem 8

Prove that $L^2(\mathbb{R})$ is separable.

Proof. The space of countable $\mathbb{Q}[i]$-linear combinations of indicators of open and/or closed intervals with endpoints in $\mathbb{Q}$ is countable and dense in $L^2(\mathbb{R})$. $\square$

Problem 9

Show that the unit ball in an infinite dimensional Hilbert space $\mathcal{H}$ contains infinitely many disjoint translates of a ball of radius $\sqrt{2}/4$. Conclude that one cannot have a nontrivial (non-zero and locally finite) translation invariant measure on an infinite dimensional Hilbert space.

Proof. Let $\{x_i\}$ be an orthonormal basis for $\mathcal{H}$. Since $\mathcal{H}$ is infinite dimensional, there must be infinitely many x_i. For convenience, let $r = \sqrt{2}/4$. We will show that the balls $B_i := B_r((1-r)x_i)$ are all disjoint. To that end, suppose $y \in B_i \cap B_j$. Write $y = \sum_k a_k x_k$. Then we have

$$|a_i - (1-r)|^2 + \sum_{k \neq i} |a_k|^2 < r^2,$$

and the similar inequality for the index j. Using $||y||^2 = \sum_k |a_k|^2$, we can rewrite the inequality as

$$||y||^2 < 2r - 1 + 2(1-r)\mathrm{Re}(a_i).$$

Now, let $a_i = a + ci$, $a_j = b + di$. Since $||y||^2 \geq |a_i|^2 + |a_j|^2 = a^2 + b^2 + c^2 + d^2 \geq a^2, b^2$, we have

$$a^2 < 2r - 1 + 2(1-r)a,$$
$$b^2 < 2r - 1 + 2(1-r)b.$$

By the quadratic formula, we find that these conditions impose $1 - 2r < a, b < 1$. But then $||y||^2 \geq a^2 + b^2 > 2(1-2r)^2$, and combining this with an earlier inequality, we get

$$2(1-2r)^2 < 2r - 1 + 2(1-r)a,$$

and similarly for b. Solving these gives

$$\frac{(3-4r)(1-2r)}{2(1-r)} < a, b.$$

At this point, we see that we can keep iterating this process to get new lower bounds for a and b. It remains to show that these bounds are increasing. Note that we have a direct formula for the new bound: If $C < a, b$, then

$$\frac{2C^2 + 1 - 2r}{2(1-r)} < a, b.$$

So we are studying the iterations of the function $f(C) = \dfrac{2C^2 + 1 - 2r}{2(1-r)}$, with starting value $C_0 = 1 - 2r$. By hand or by calculator, we can see that $C_6 > 1$.

But earlier we established that $a, b < 1$, so we have a contradiction! This shows that $B_i \cap B_j = \varnothing$. Since there is one ball for each basis element, and we have infinitely many basis elements, we are done.

Now, we haven't yet shown that each B_i is contained in the unit ball $B_1(0)$, so let us do that now. In the above work, we showed that if $y \in B_i$, then $||y||^2 < 2r - 1 + 2(1-r)a$ for some real number a satisfying $a < 1$. Thus $||y||^2 < 2r - 1 + 2(1-r) = 1$ as desired.

If one had a locally finite translation invariant measure on $\mathcal{H}$, then for $r > 0$ small enough, $B_r(0)$ would have to be finite measure. However, by the above result, r would contain infinitely many copies of $B_{r\sqrt{2}/4}(0)$, contradicting finiteness unless the balls had measure 0, which would imply $\mathcal{H}$ has measure 0 by upper continuity of measures. $\square$

Problem 10

Suppose Ω is a compact metric space with metric ρ and some measure μ. Let T be a measure-preserving ergodic transformation with the additional property that $\rho(Tx, Ty) = \rho(x, y)$. Show that if f is a continuous function on Ω, then $(1/N)\sum_{n=0}^{N-1} f(T^n w)$ converges uniformly to $\int_\Omega f d\mu$.
Hint: mean ergodic and Ascoli theorems.

Proof. We first show that the family of functions $M_N f$ defined by

$$(M_N f)(w) = \frac{1}{N}\sum_{n=0}^{N-1} f(T^n w)$$

is uniformly equicontinuous. Fix $w' \in \Omega$ let $\varepsilon > 0$. We want to show that there is $\delta > 0$ such that for all $w \in \Omega$ and all $N > 0$, we have $\rho(w, w') < \delta$ implies $|(M_N f)(w) - (M_N f)(w')| < \varepsilon$. This follows from a kind of $\varepsilon/3$ argument using all of our hypotheses.

Since the family of functions $M_N f$ is uniformly equicontinuous, to show that $M_n f \to \int_\Omega f d\mu$ uniformly is equivalent to showing it pointwise. □

Problem 11

Let $\{x_\alpha\}_{\alpha\in A}$ be an orthonormal basis for a Hilbert space. Let $\sum_{\alpha\in A} |c_\alpha|^2 < \infty$. Prove that $\sum_{\alpha\in A} c_\alpha x_\alpha$ converges and is independent of the order of summation.

Proof. Note to reader: I will use "countable" for "at most countable". Since $\sum_{\alpha\in A} |c_\alpha|^2 < \infty$, the set $B = \{\alpha \in A \mid c_\alpha \neq 0\}$ is countable. Indeed, if it wasn't, then one of the sets $B_n = \{\alpha \in A \mid |c_\alpha|^2 > \frac{1}{n}\}$ would be uncountable, since if they were all countable, we would have that B is countable union of countable sets, hence countable. If B_n is uncountable, then

$$\sum_{\alpha\in A} |c_\alpha|^2 \geq \sum_{\alpha\in B_n} |c_\alpha|^2 > \sum_{j=1}^{\infty} \frac{1}{n} = \infty,$$

a contradiction. Thus B is countable. Choose two orderings $\alpha_1, \ldots,$ and $\beta_1, \ldots,$ for B. So that we don't have to deal with the cases of B being finite and infinite separately, we will treat the infinite case and say that if B has only n elements, that $c_{\alpha_i} = c_{\beta_i} = 0$ for $i > n$. Now, since the summands of $\sum_{\alpha\in A} |c_\alpha|^2$ are non-negative, the sum is independent of the ordering. In particular, the sequence $\sum_{j=1}^{n} |c_{\alpha_j}|^2$ converges, so that the differences $\sum_{j=m}^{n} |c_{\alpha_j}|^2$ can be made arbitrarily small for sufficiently large $m < n$. Now, consider the sequence $\sum_{j=1}^{n} c_{\alpha_j} x_{\alpha_j}$. By the Pythagorean theorem, the norm squared of the differences $\sum_{j=m}^{n} c_{\alpha_j} x_{\alpha_j}$ is $\sum_{j=m}^{n} |c_{\alpha_j}|^2$, which can be made arbitrarily small as previously noted. Thus the sequence $\sum_{j=1}^{n} c_{\alpha_j} x_{\alpha_j}$ is Cauchy, hence converges to some value x.

By the same argument, the sequence $\sum_{j=1}^{n} c_{\beta_j} x_{\beta_j}$ converges to some element y. We want to show $x = y$. By linearity and continuity of the inner product, we see that for all $\alpha \in A$, $(x_\alpha, x) = c_\alpha = (x_\alpha, y)$. By the proof of Theorem II.6 in the book, we then have $x = \sum_{\alpha\in A} c_\alpha x_\alpha = y$, no matter what order is used to define the sum in the middle. $\square$