MATH 7330 Homework 1

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Problem 1

Let (X, d) be a metric space. Prove the following:

(a) A set O is open iff $X \setminus O$ is closed.

(b) $x_m \xrightarrow{d} x$ iff for each neighborhood N of x, there exists an M so that $m \geq M$ implies $x_m \in N$.

(c) The set of interior points of a set is open.

(d) The union of a set E with its limit points is a closed set.

(e) A set is open if and only if it is a neighborhood of each of its points.

Proof. (a) Let O be open. Let x be a limit point of $X \setminus O$. Suppose $x \in O$. Then there is $r > 0$ such that $B(x; r) \subset O$. Then $B(x; r) \cap (X \setminus O) = \varnothing$, which contradicts the hypothesis that x is a limit point of $X \setminus O$. Thus $x \notin O$, i.e. $x \in X \setminus O$. Since x was an arbitrary limit point of $X \setminus O$, it follows that $X \setminus O$ contains all of its limit points, so it is closed.

Conversely, let $X \setminus O$ be closed. Let $x \in O$. Since $x \notin X \setminus O$ and $X \setminus O$ contains its limit points, x is not a limit point of $X \setminus O$. Therefore, there is some $r > 0$ such that $B(x; r) \cap ((X \setminus O) \setminus \{x\}) = \varnothing$. Since $x \notin X \setminus O$, we have $(X \setminus O) \setminus \{x\} = X \setminus O$. Thus $B(x; r) \cap (X \setminus O) = \varnothing$. This is equivalent to $B(x; r) \subset O$. Since x is an arbitrary point in O, this means O is open.

(b) Suppose $x_m \xrightarrow{d} x$. Let N be a neighborhood of x. Let $r > 0$ be such that $B(x; r) \subset N$. Since $\lim\limits_{n\to\infty} d(x_n, x) = 0$, there is an M such that $m \geq M$ implies $d(x_m, x) < r$. If $d(x_m, x) < r$, then $x_m \in B(x; r)$ by definition. Since $B(x; r) \subset N$, this means $x_m \in N$. Putting these implications together, we have that $m \geq M$ implies $x_m \in N$.

Conversely, suppose that for each neighborhood N of x, there exists an M so that $m \geq M$ implies $x_m \in N$. Clearly, for all $r > 0$, the ball $B(x; r)$ is a neighborhood of x. Thus, for all $r > 0$, there is an M such that $m \geq M$ implies $x_m \in B(x;r)$, or equivalently, $d(x_m, x) < r$. Since $d(x_n, x) \geq 0$, we have $\lim_{n\to\infty} d(x_n, x) = 0$. Thus $x_n \xrightarrow{d} x$.

(c) Let Y be a subset of X, and let O be the set of interior points of Y. Let $x \in O$. Then for some $r > 0$, we have $B(x;r) \subset Y$. For all $y \in B(x;r)$, we have $B(y; r - d(x,y)) \subset B(x;r)$ by the triangle inequality. Then $B(y; r - d(x,y)) \subset Y$, which means y is an interior point of Y, i.e. $y \in O$. Since y is an arbitrary point in $B(x;r)$, this means $B(x;r) \subset O$. Since x is an arbitrary point in O, this means O is open.

(d) Let L be the set of limit points of E, and let $\overline{E} = E \cup L$. Let x be a limit point of $\overline{E}$. Suppose that $x \notin \overline{E}$, or equivalently $x \notin E$ and $x \notin L$. Then there is some $r > 0$ such that $B(x;r) \cap E = \varnothing$. On the other hand, we know that $B(x;r) \cap \overline{E} \neq \varnothing$, so we must have $B(x;r) \cap L \neq \varnothing$. Let $y \in B(x;r) \cap L$. Since $B(x;r) \cap E = \varnothing$, we know $y \notin E$. Since $y \in L$, we know that for all $t > 0$, $B(y;t) \cap E \neq \varnothing$. Since $y \in B(x;r)$, we have $d(x,y) < r$. Choosing $t = r - d(x,y)$, we have $B(y;t) \subset B(x;r)$ by the triangle inequality, and $B(y;t) \cap E \neq \varnothing$ from before. However, these two facts imply $B(x;r) \cap E \neq \varnothing$, which contradicts the previously established $B(x;r) \cap E = \varnothing$. Therefore, the assumption that $x \notin \overline{E}$ must be false. Therefore, $\overline{E}$ has all of its limit points, meaning it is closed.

(e) Let O be open. Let $y \in O$. By definition, there exists $r > 0$ such that $B(y;r) \subset O$. By definition, this means O is a neighborhood of y. Since y was chosen arbitrarily, this means O is a neighborhood of all of its points.

Conversely, suppose O is a neighborhood of all of its points. Let $y \in O$. Then O is a neighborhood of y, so $B(y;r) \subset N$ for some $r > 0$. Since y was chosen arbitrarily, this means O is open by definition.

$\square$

Problem 2

A function $f : (X, d) \to (Y, \rho)$ between metric spaces is continuous iff for all open sets $O \subset Y$, $f^{-1}(O)$ is open.

Proof. Let f be continuous. Let $O \subset Y$ be open. Let $x \in f^{-1}(O)$. We want to show that there is $r > 0$ such that $B(x; r) \subset f^{-1}(O)$. Equivalently, we want to show that there is $r > 0$ such that $d(x, y) < r$ implies $f(y) \in O$. Start with $r = 1$. If $B(x; 1) \subset f^{-1}(O)$, then we are done. Otherwise, let $x_1 \in B(x; 1) \setminus f^{-1}(O)$. Then take $r = \frac{1}{2}, \frac{1}{3}$, and so on. If for some n we have $B\left(x; \frac{1}{n}\right) \subset f^{-1}(O)$, then we are done. Otherwise, we have a sequence $x_1, x_2, \ldots$ with $x_n \in B\left(x; \frac{1}{n}\right)$, i.e. $d(x_n, x) < \frac{1}{n}$. Since $\lim_{n\to\infty} \frac{1}{n} = 0$, we have $x_n \xrightarrow{d} x$. Since f is continuous, we have $f(x_n) \xrightarrow{\rho} f(x)$. By parts (b) and (e) of Theorem I.4, there exists an M such that $m \geq M$ implies $f(x_m) \in O$. However, we assumed that $f(x_n) \notin O$ for all n, so we have a contradiction. It follows that for some n, $B\left(x; \frac{1}{n}\right) \subset f^{-1}(O)$, so $f^{-1}(O)$ is open.

Conversely, suppose that for all open sets $O \subset Y$, we have $f^{-1}(O)$ is open. Let $x_n \xrightarrow{d} x$. Let N be a neighborhood of $f(x)$. In particular, let $r > 0$ be such that $B(f(x); r) \subset N$. Since $B(f(x); r)$ is open, $f^{-1}(B(f(x); r))$ is open. Since $f(x) \in B(f(x); r)$, we have $x \in f^{-1}(B(f(x); r))$. By parts (b) and (e) of Theorem I.4, there exists an M such that $m \geq M$ implies $x_m \in f^{-1}(B(f(x); r))$, which implies $f(x_m) \in B(f(x); r)$, which implies $f(x_m) \in N$. Since N is an arbitrary neighborhood of $f(x)$, we have that $f(x_n) \xrightarrow{\rho} f(x)$ by part (b) of Theorem I.4. $\square$

Problem 3

Let $T : X \to Y$ be a linear transformation between normed linear spaces $(X, ||\cdot||_X)$ and $(Y, ||\cdot||_Y)$. The following are equivalent:

(a) T is continuous at one point.

(b) T is continuous at all points.

(c) T is bounded.

Proof. Suppose T is continuous at some $x \in X$. Let $x' \in X$ and let $x'_n \in X$ converge to x'. Let $x_n = x - x' + x'_n$. Then $||x_n - x||_X = ||x'_n - x'||_X$ can be made arbitrarily small for sufficiently large n, i.e. x_n converges to x. Thus $Tx_n = Tx - Tx' + Tx'_n$ converges to Tx. In particular, $||Tx - Tx' + Tx'_n - Tx||_X = ||Tx'_n - Tx'||_X$ can be made arbitrarily small for sufficiently large n, i.e. Tx'_n converges to Tx'. Thus T is continuous at x', which was an arbitrary point in X. Thus (a) implies (b).

Now suppose T is continuous at all points. Continuity at 0 implies that there is a $\delta > 0$ such that $||x||_X < \delta$ implies $||Tx||_Y < 1$. Let δ' be any positive real less than δ. Then, for any non-zero x, the vector $x' = \delta' x / ||x||_X$ satisfies $||x'||_X < \delta$. Therefore, $||Tx'||_Y < 1$, which can be rewritten as $||Tx||_Y < ||x||_X/\delta'$. This trivially implies $||Tx||_Y \leq ||x||_X/\delta'$, which also holds for $x = 0$, since both sides become 0. We can take $C = 1/\delta'$ to get that T is bounded. Thus (b) implies (c).

Finally, suppose T is bounded, with bound C. If $C = 0$, then $||Tx|| \leq 0$, so $Tx = 0$ for all x. Clearly the zero function is continuous. Then we can suppose C is not zero and continue. Let $x_n \in X$ converge to 0. Since $T0 = 0$, we must also show that Tx_n converges to 0. Let $\varepsilon > 0$, and let n be large enough so that $||x_n||_X < \varepsilon/C$. Then $||Tx_n||_Y \leq C||x_n|| < \varepsilon$, so Tx_n converges to 0, as desired. Thus (c) implies (a). □

Problem 4

Suppose T is a bounded linear transformation from a normed linear space $(V_1, ||\cdot||_1)$ to a complete normed linear space $(V_2, ||\cdot||_2)$. Then T can be uniquely extended to a bounded linear transformation (with the same bound), $\tilde{T}$, from the completion of V_1 to V_2.

Proof. Let $\tilde{V}_1$ be the completion of V. We will also use $||\cdot||_1$ to denote the norm on $\tilde{V}_1$. The book proves the existence of $\tilde{T}$. Namely, for $x \in \tilde{V}_1$ and $x_n \in V_1$ converging to x, $\tilde{T}x$ is defined as the limit of Tx_n.

For $x \in V_1$, considered as an element of $\tilde{V}_1$, we can take the constant sequence $x_n = x$ which obviously converges to x, so that $\tilde{T}x = \lim Tx_n = \lim Tx = Tx$. Thus $\tilde{T}$ extends T.

Now, suppose $||Tx||_2 \leq C||x||_1$ for all $x \in V_1$. Pick $x \in \tilde{V}_1$ and a sequence x_n in V_1 converging to x. The first claim the book makes without proof is that $||\tilde{T}x||_2 = \lim\limits_{n\to\infty} ||Tx_n||_2$. First, we show a stronger result: $d(\lim\limits_{n\to\infty} y_n, z) = \lim\limits_{n\to\infty} d(y_n, z)$ in any metric space, assuming $\lim\limits_{n\to\infty} y_n$ exists.

Towards this goal, let $y = \lim\limits_{n\to\infty} y_n$ and let $\varepsilon > 0$. Pick N so that $n \geq N$ implies $d(y_n, y) < \varepsilon$. Then $d(y_n, z) \leq d(y_n, y) + d(y, z) < d(y, z) + \varepsilon$ and $d(y, z) \leq d(y, y_n) + d(y_n, z) < \varepsilon + d(y_n, z)$. Rewriting these gives $d(y_n, z) - d(y, z) < \varepsilon$ and $d(y, z) - d(y_n, z) < \varepsilon$, which implies $|d(y, z) - d(y_n, z)| < \varepsilon$. This shows that $\lim\limits_{n\to\infty} d(y_n, z) = d(y, z)$ as desired.

To apply this result to our problem, recall that a norm induces a metric by $d(y, z) = ||y - z||$, and that $||y|| = d(y, 0)$. In our case, we have

$$\begin{aligned} ||\tilde{T}x||_2 &= ||\lim_{n\to\infty} Tx_n||_2 = d(\lim_{n\to\infty} Tx_n, 0) \\ &= \lim_{n\to\infty} d(Tx_n, 0) = \lim_{n\to\infty} ||Tx_n||_2. \end{aligned}$$

Now I will deviate from the book. First, note that since x_n converges to x, the norms $||x_n||_1$ converge to $||x||$. Indeed, by the triangle inequality, $||x - x_n|| \geq |||x|| - ||x_n|||$, and the left hand side can be made arbitrarily small. Thus $\lim\limits_{n\to\infty} C||x_n||_1$ exists and is equal to $C||x||_1$. For notational simplicity, let $a_n = ||Tx_n||_2$, $a = \lim\limits_{n\to\infty} a_n$, $b_n = C||x_n||_1$, $b = \lim\limits_{n\to\infty} b_n$. We will show $a \leq b$. By hypothesis on T, we know $a_n \leq b_n$ for all n. Suppose $a > b$. Let $\varepsilon = \frac{1}{2}(a - b)$. Let n be large enough so that $|a - a_n| < \varepsilon$ and $|b - b_n| < \varepsilon$. Then $b_n < b + \varepsilon = a - \varepsilon < a_n$, which contradicts $a_n \leq b_n$. Thus we must have $a \leq b$. In our case, we have shown $\lim\limits_{n\to\infty} ||Tx_n||_2 \leq \lim\limits_{n\to\infty} C||x_n||_1$. We also know that $\lim\limits_{n\to\infty} ||Tx_n||_2 = ||\tilde{T}x||_2$ and $\lim\limits_{n\to\infty} C||x_n||_1 = C||x||_1$. Thus $||\tilde{T}x||_2 \leq C||x||_1$, meaning $\tilde{T}$ is bounded with the same bound as T.

Let $x, y \in \tilde{V}_1$ and let α be a scalar. Let x_n and y_n be sequences in V_1 converging to x and y. By definition, $\alpha x + y$ is $\lim_{n\to\infty}(\alpha x_n + y_n)$. Thus,

$$\tilde{T}(\alpha x + y) = \lim_{n\to\infty} T(\alpha x_n + y_n)$$
$$= \alpha \lim_{n\to\infty}(Tx_n) + \lim_{n\to\infty}(Ty_n) = \alpha Tx + Ty,$$

so T is linear.

Finally, we will show that $\tilde{T}$ is the unique extension of T to a bounded linear transformation from $\tilde{V}_1$ to V_2. Indeed, suppose J is another extension. Since J is bounded, it is continuous, by Theorem I.6. Let $x \in \tilde{V}_1$ and let $x_n \in V_1$ converge to x. Then Jx_n converges to Jx, but by hypothesis, $Jx_n = Tx_n$, and by definition, Tx_n converges to $\tilde{T}x$. Thus $Jx = \tilde{T}x$, so $\tilde{T}$ is the unique extension.

□

Problem 5

If $f_n \to f$ in L^1, then some subsequence f_{n_i} converges pointwise a.e. to f.

Proof. Convergent sequences are Cauchy, so we can pick a subsequence $g_i = f_{n_i}$ with $||g_i - g_{i+1}||_1 \leq 2^{-i}$. Let

$$h_m(x) = \sum_{i=1}^{m} |g_i(x) - g_{i+1}(x)|$$

and let h_∞ be the infinite sum. We have $h_m \nearrow h_\infty$ and $\int |h_m| \leq \sum_{i=1}^{m} ||g_i - g_{i+1}||_1 \leq 1$, so $h_\infty \in L^1$ by monotone convergence. In particular, $h_\infty(x) < \infty$ a.e. Note that we can write

$$g_m(x) = g_1(x) - \sum_{i=1}^{m-1} (g_i(x) - g_{i+1}(x)).$$

In particular, since

$$\left| \sum_{i=1}^{m-1} (g_i(x) - g_{i+1}(x)) \right| \leq h_{m-1}(x) \leq h_\infty(x)$$

is finite a.e., we have that g_m converges pointwise a.e. to a function, call it g. By the above argument, we have $|g_m(x)| \leq |g_1(x)| + h_\infty(x)$, and the right hand side is a fixed function in L^1, so $g \in L^1$ and $g_m \to g$ in L^1 by dominated convergence. But g_m is a subsequence of f_n, which we know converges to f in L^1. In particular, we must have $f = g$. Furthermore, we showed that g_m converges pointwise a.e. to g; in other words, we showed that a subsequence of f_n converges pointwise a.e. to f. $\square$

Problem 6

(a) Prove that for any open set A in $[0,1]$, χ_A is an L^1 limit of continuous functions.

(b) Let B be a Borel set in $[0,1]$. Prove that χ_B is an L^1 limit of functions χ_A with A open.

(c) Prove $C[a,b]$ is L^1 dense in $L^1[a,b]$.

Proof. (a) Write A as a countable union of pairwise disjoint open intervals (a_i, b_i), ordered so that $b_i \leq a_{i+1}$, and possibly the sets $\{0\}$ and $\{1\}$ if $a_1 = 0$ or $\max b_i = 1$ respectively. Define continuous functions f_n as follows. If $x \in A$, then $f_n(x) = 1$. Furthermore, set $f_n(a_i) = f_n(b_i) = 1$ for all i. If $b_i \neq a_{i+1}$, then set

$$f_n(x) = \begin{cases} \frac{-2n}{a_{i+1}-b_i}\left(x - b_i - \frac{a_{i+1}-b_i}{2n}\right) & b_i \leq x \leq b_i + \frac{a_{i+1}-b_i}{2n}, \\ 0 & b_i + \frac{a_{i+1}-b_i}{2n} \leq x \leq a_{i+1} - \frac{a_{i+1}-b_i}{2n}, \\ \frac{2n}{a_{i+1}-b_i}\left(x - a_{i+1} + \frac{a_{i+1}-b_i}{2n}\right) & a_{i+1} - \frac{a_{i+1}-b_i}{2n} \leq x \leq a_{i+1}. \end{cases}$$

Intuitively, we are defining $f_n(x)$ to drop down to zero and come back up, but faster and faster as n gets bigger. If 0 is not in A, or if $\sup b_i \neq 1$, then we also need to do a similar thing between 0 and a_1, and between $\sup b_i$ and 1. I will not write that out explicitly. If $\sup b_i = 1$, we take $f_n(1) = 1$.

Now, we claim that the f_n converge in L^1 to χ_A. The only contributions to $||f_n - \chi_A||_1$ will be the triangles when $f_n(x)$ jumps down between intervals. Explicitly, the contribution between b_i and a_{i+1} is $\dfrac{a_{i+1} - b_i}{2n}$. Since $A \subset [0,1]$, we have $\sum\limits_{i=1}^{\infty}(a_{i+1} - b_i) \leq 1$, so the contributions to $||f_n - \chi_A||_1$ between the intervals is bounded by $\dfrac{1}{2n}$, which goes to 0. There are also contributions in the cases $a_1 \neq 0$ and $\sup b_i \neq 1$ which will be triangles with fixed height and a base length that goes to 0. In all, we have $||f_n - \chi_A||_1 \to 0$, as desired.

(b) Note that $||\chi_B||_1$ is the Lebesgue measure of B, which is the infimum of the Lebesgue measures of open sets A containing B. In particular, there is a sequence of A_n of open sets that contain B such that $\mu(B) = \lim\limits_{n\to\infty} \mu(A_n)$. Note that μ does not take infinite values on $[0,1]$, since for all measurable M, $\mu(M) \leq \mu([0,1]) = 1$. Now, $\mu(B) = \mu(B \setminus A_n) + \mu(A_n)$, and since everything is finite, we can write $\mu(B \setminus A_n) = \mu(B) - \mu(A_n)$. Since $\mu(B) = \lim\limits_{n\to\infty} \mu(A_n)$, we have $\lim\limits_{n\to\infty} \mu(B \setminus A_n) = 0$. We can write $\chi_{B\setminus A_n} = \chi_B - \chi_{A_n}$, so we have $\lim\limits_{n\to\infty} ||\chi_B - \chi_{A_n}||_1 = \mu(B \setminus A_n)$ converging to 0. In other words, χ_B is the L^1 limit of χ_{A_n}.

(c) From parts (a) and (b), and since limits are linear, it suffices to show that any $f \in L^1[a,b]$ is an L^1 limit of functions of the form $\sum_{m=1}^{n} c_m \chi_{B_m}$ with B_m pairwise disjoint Borel sets. In fact, since $f \in L^1$ is a difference of non-negative L^1 functions, namely $\max(f,0)$ and $\max(-f,0)$, we can assume $f \geq 0$ and $c_m \geq 0$.

First, a lemma. Suppose an L^1 function ϕ on $[0,1]$ takes finitely many values, say $c_1, \ldots, c_n$. Then we can write $\phi = \sum_{m=1}^{n} c_m \chi_{B_m}$, where each $B_m = \phi^{-1}(\{c_m\})$ is Borel and disjoint from the other B_i. There is nothing involved in the proof of this fact; it is true by definitions.

Now for $n \geq 1$, define

$$f_n(x) = \min\left(n, \frac{\lfloor 2^n f(x) \rfloor}{2^n}\right).$$

We have $f_n \in L^1$ by basic properties of L^1. Furthermore, f_n takes finitely many values: $0, \frac{1}{2^n}, \ldots, n$. Thus, we can write f_n as a finite linear combination of indicators of Borel sets by our lemma.

We claim that $f_n \to f$ in L^1. We will use monotone convergence. First we show $f_n(x) \leq f_{n+1}(x)$. There are three cases to consider:

In the first case, $f_{n+1}(x) = n+1$, in which case the inequality is obvious since $f_n(x) \leq n$ by definition.

In the second case, $f_n(x) = 2^{-n}\lfloor 2^n f(x) \rfloor$ and $f_{n+1}(x) = 2^{-n-1}\lfloor 2^{n+1} f(x) \rfloor$. In this case, the inequality is a special case of the inequality $2\lfloor a \rfloor \leq \lfloor 2a \rfloor$, which is also proved by cases: If $a \in [k, k+\frac{1}{2})$ for $k \in \mathbb{Z}$, then $2\lfloor a \rfloor = 2k$ and $\lfloor 2a \rfloor = 2k$; If $a \in [k+\frac{1}{2}, k+1)$ for $k \in \mathbb{Z}$, then $2\lfloor a \rfloor = 2k$ and $\lfloor 2a \rfloor = 2k+1$.

In the last case, $f_n(x) = n$ and $f_{n+1}(x) = 2^{-n-1}\lfloor 2^{n+1} f(x) \rfloor$. Note that $f_n(x) = n$ implies $n \leq 2^{-n}\lfloor 2^n f(x) \rfloor$, and by the previous case, we know $2^{-n}\lfloor 2^n f(x) \rfloor \leq 2^{-n-1}\lfloor 2^{n+1} f(x) \rfloor$, so together we get the desired inequality.

We have shown that the f_n are monotone increasing, non-negative L^1 functions. It remains to show that they converge pointwise a.e. to f. Since $f \in L^1$, f is finite a.e. Let x be any point with $f(x) < \infty$. Then for all $n \geq f(x)$, we have $2^{-n}\lfloor 2^n f(x) \rfloor \leq 2^{-n}\lfloor n 2^n \rfloor = n$, so $f_n(x) = 2^{-n}\lfloor 2^n f(x) \rfloor$. Now, since $a - \lfloor a \rfloor < 1$ for all a, we have $f(x) - f_n(x) < 2^{-n}$, so $f_n(x)$ converges pointwise a.e. to $f(x)$. Thus, by monotone convergence theorem, the f_n converge to f in L^1.

To recap, we have shown that any $f \in L^1$ is an L^1 limit of finite linear combinations of indicators of Borel sets. Since the indicators are L^1 limits of continuous functions by parts (a) and (b), it follows that f is an L^1 limit of continuous functions.

$\square$

Problem 7

Use an $\varepsilon/3$ argument to prove the following: Let B be a complete normed linear space and suppose T_n is a sequence of linear maps $T_n : B \to B$ with two properties:

(i) The T_n are bounded uniformly in n; $||T_n|| \leq C$ for all n.

(ii) For a dense set $D \subset B$, $T_n x$ converges if $x \in D$.

Show that $T_n x$ converges for each x and that the limiting function Tx is a bounded linear map.

Proof. Let C be the bound given in condition (i). If $C = 0$, then each T_n must be the zero map, in which case $T_n x = 0$ obviously converges for all x. Thus we will assume $C > 0$. Since B is complete, we will show that $T_n x$ is a Cauchy sequence, and this will imply that $T_n x$ converges.

Let $\varepsilon > 0$ and let $x \in B$. Let (x_k) be a sequence of elements in D which converge to x. Fix k large so that $||x - x_k|| < \varepsilon/(3C)$. Since $T_n x$ converges, it is Cauchy; fix N so that $m, n \geq N$ implies $||T_n x_k - T_m x_k|| < \varepsilon/3$. Then for $m, n \geq N$, we have

$$\begin{aligned}
||T_n x - T_m x|| &\leq ||T_n x - T_n x_k|| + ||T_n x_k - T_m x_k|| + ||T_m x_k - T_m x|| \\
&= ||T_n(x - x_k)|| + ||T_n x_k - T_m x_k|| + ||T_m(x_k - x)|| \\
&\leq C||x - x_k|| + ||T_n x_k - T_m x_k|| + C||x_k - x|| \\
&< C\varepsilon/(3C) + \varepsilon/3 + C\varepsilon/(3C) = \varepsilon.
\end{aligned}$$

Therefore, we have a well-defined function $T : B \to B$. We must now show it is bounded and linear.

Let $\varepsilon > 0$, and let n be large enough so that $||Tx - T_n x|| < \varepsilon$. Then $||Tx|| \leq ||Tx - T_n x|| + ||T_n x|| < \varepsilon + C||x||$. Since $||Tx|| - C||x|| < \varepsilon$ for all $\varepsilon > 0$, we must have $||Tx|| - C||x|| \leq 0$, so T is bounded.

Now let $x, y \in B$ and α a scalar. We want to show $T(\alpha x + y) = \alpha Tx + Ty$. This is trivial if $\alpha = 0$, so assume $\alpha \neq 0$. Since T_n converges to T pointwise, we can choose n large enough so that we have the following three inequalities:

$$\begin{aligned}
||T(\alpha x + y) - T_n(\alpha x + y)|| &< \varepsilon/3, \\
||T_n x - Tx|| &< \varepsilon/(3|\alpha|), \\
||T_n y - Ty|| &< \varepsilon/3.
\end{aligned}$$

Then we have

$$\begin{aligned}
&||T(\alpha x + y) - (\alpha Tx + Ty)|| \\
&\leq ||T(\alpha x + y) - T_n(\alpha x + y)|| + ||T_n(\alpha x + y) - (\alpha Tx + Ty)|| \\
&= ||T(\alpha x + y) - T_n(\alpha x + y)|| + ||\alpha T_n x + T_n y - (\alpha Tx + Ty)|| \\
&\leq ||T(\alpha x + y) - T_n(\alpha x + y)|| + ||\alpha T_n x - \alpha Tx|| + ||T_n y - Ty|| \\
&= ||T(\alpha x + y) - T_n(\alpha x + y)|| + |\alpha|||T_n x - Tx|| + ||T_n y - Ty|| \\
&< \varepsilon/3 + |\alpha|\varepsilon/(3|\alpha|) + \varepsilon/3 = \varepsilon.
\end{aligned}$$

The only way for $||A|| < \varepsilon$ for all $\varepsilon > 0$ is if $||A|| = 0$, which means $A = 0$. Hence we have $T(\alpha x + y) - (\alpha Tx + Ty) = 0$, so T is linear. $\square$

Problem 8

Use an $\varepsilon/3$ argument to prove the following: Let $\{f_n\}$ be an equicontinuous family of functions from metric space (X, d) to metric space (Y, ρ), with Y complete. Suppose that for a dense set $D \subset X$, we know $f_n(x)$ converges for all $x \in D$. Then show that $f_n(x)$ converges for all x.

Proof. We will show that for all x, $f_n(x)$ is a Cauchy sequence. Since Y is complete, this will imply $f_n(x)$ converges.

Let $\varepsilon > 0$ and let $x \in X$. By equicontinuity, let $\delta > 0$ be such that for all n, $d(x, x') < \delta$ implies $\rho(f_n(x), f_n(x')) < \varepsilon/3$. Next, let (x_k) be a sequence of elements in D that converges to x. Fix k large so that $d(x, x_k) < \delta$; this implies $\rho(f_n(x), f_n(x_k)) < \varepsilon/3$ for all n. Since $f_n(x_k)$ converges, it is Cauchy; let N be such that $n, m \geq N$ implies $\rho(f_n(x_k), f_m(x_k)) < \varepsilon/3$. Then for $n, m \geq N$, we have

$$\begin{aligned}\rho(f_n(x), f_m(x)) &\leq \rho(f_n(x), f_n(x_k)) + \rho(f_n(x_k), f_m(x_k)) + \rho(f_m(x_k), f_m(x)) \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

□

Problem 9

(a) Using the Heine-Borel property, prove that a continuous function on $[0,1]$ is uniformly continuous.

(b) Prove that an equicontinuous family of functions on $[0,1]$ is uniformly equicontinuous.

Proof. (a) Since $[0,1]$ is closed and bounded, it satisfies the property that any open cover has a finite subcover. Let $f \in C[0,1]$ and let $\varepsilon > 0$. For each $x \in [0,1]$, there is a $\delta_x > 0$ so that if $x' \in [0,1]$ and $|x - x'| < \delta_x$, then $|f(x) - f(x')| < \varepsilon/2$. The open balls of radius $\delta_x/2$ at each x give an open cover of $[0,1]$. There is a finite subcover, say given by balls of radius $\delta_1/2, \ldots, \delta_n/2$ at $x_1, \ldots, x_n$. Let δ be the minumum of the $\delta_1/2, \ldots, \delta_n/2$.

Now let $x, y \in [0,1]$ with $|x-y| < \delta$. There is some i such that $|x-x_i| < \delta_i/2$. Then

$$|y - x_i| \leq |x - y| + |x - x_i| < \delta + \delta_i/2 \leq \delta_i.$$

By definition of the δ_i, it follows that

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_i)| + |f(x_i) - f(y)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

(b) Let the family be $\mathcal{F}$. Let $\varepsilon > 0$. We will construct a number δ depending only on ε as follows. Since $\mathcal{F}$ is equicontinuous, for all $x \in [0,1]$, there is $\delta_x > 0$ such that for all $f \in \mathcal{F}$, we have that $|x - x'| < \delta_x$ implies $|f(x) - f(x')| < \varepsilon/2$. The open balls of radius $\delta_x/2$ at x give an open cover, so we can choose a finite subcover by the Heine-Borel property of $[0,1]$. Let the finite subcover consist of balls of radius $\delta_1/2, \ldots, \delta_n/2$ at points $x_1, \ldots, x_n$. Let δ be the minimum of the $\delta_i/2$.

Now let $|x - y| < \delta$. There is some i such that $|x - x_i| < \delta_i/2$. Then

$$\begin{aligned} |y - x_i| &\leq |y - x| + |x - x_i| \\ &< \delta + \delta_i/2 \leq \delta_i. \end{aligned}$$

For all $f \in F$ we now have

$$\begin{aligned} |f(x) - f(y)| &\leq |f(x) - f(x_i)| + |f(x_i) - f(y)| \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

This proves $\mathcal{F}$ is uniformly equicontinuous.

□