MATH 7311 Homework 8

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1 Problem 1

Show that $\lim_{n\to\infty} \int_{[1,\infty)} \sin(x/n)\frac{n^3}{1+n^2x^3} d\lambda$ exists and then find the limit.

Proof. Recall the limit $\lim_{x\to 0}\sin(x)/x = 1$. Then $\lim_{n\to\infty}\sin(x/n)n = x$ for fixed x (special care should be taken for $x = 0$, but in this case we get $\sin(x/n)n = 0$ for all n). Thus, $\sin(x/n)\frac{n^3}{1+n^2x^3} \to \frac{1}{x^2}$ pointwise, which is integrable over $[1,\infty)$. Now note that $|\sin(x)| \leq |x|$ for all x (this can be proven with a simple geometry argument). Thus $|\sin(x/n)\frac{n^3}{1+n^2x^3}| \leq \frac{n^2x}{1+n^2x^3} \leq \frac{1}{x^2}$ for $x > 0$. Thus the sequence is dominated by an integrable function, and we can apply LDCT to get $\lim_{n\to\infty}\int_{[1,\infty)}\sin(x/n)\frac{n^3}{1+n^2x^3}d\lambda = \int_{[1,\infty)}\frac{1}{x^2}d\lambda = 1$. □

2 Problem 2

Let $f \in \mathcal{L}^1(\mathbb{R})$ such that $g(x) = xf(x)$ is integrable. Define $F(t) = \int_{\mathbb{R}} \cos(xt) f(x) d\lambda$. Show that F is continuously differentiable.

Proof. First note that $|\cos(xt)f(x)| \leq |f(x)|$, so $\cos(xt)f(x) \in \mathcal{L}^1$ for all $t \in \mathbb{R}$. Thus $F(t)$ exists. Now, $\frac{1}{h}(F(t+h)-F(t)) = \int \frac{1}{h}(\cos(x(t+h))-\cos(xt))f(x)d\lambda$. Let h_n be a sequence of reals converging to 0. Then $\lim_{n\to\infty} \frac{1}{h_n}(\cos(x(t+h_n))-\cos(xt)) = -x\sin(xt)$. Since this sequence converges, it must be bounded by a constant M. Then $|\frac{1}{h_n}(\cos(x(t+h_n))-\cos(xt))f(x)| \leq M|f(x)|$ for all n, so that the sequence of functions is dominated by $M|f|$, which is integrable. We can apply the LDCT to get that $\lim_{h\to 0} \frac{1}{h}(F(t+h)-F(t)) = \lim_{n\to\infty} \frac{1}{h_n}(F(t+h_n)-F(t)) = \lim_{n\to\infty} \int \frac{1}{h_n}(\cos(x(t+h_n)) - \cos(xt))f(x)d\lambda = \int -x\sin(xt)f(x)d\lambda$. This integral is well defined since $|-x\sin(xt)f(x)| \leq |xf(x)|$, and $xf(x)$ is integrable.

It remains to show F' is continuous. Let h_n be a sequence of reals converging to 0. Let $g_n(x) = -x\sin(x(t+h_n))f(x)$ for fixed t. Pointwise, $g_n(x) \to -x\sin(xt)f(x)$. Also, $|g_n(x)| \leq |xf(x)| = |g(x)|$, so the sequence is dominated by $|g|$. We then apply LDCT to get $F'(t+h_n) = \int g_n(x)d\lambda \to \int -x\sin(xt)f(x)d\lambda = F'(t)$, so F' is continuous as desired. $\square$

3 Problem 3

Let X be a σ-finite measure space. Let $f \in \mathcal{L}^1(X)$.

a) Let $A_n \in \mathcal{A}$ be a sequence such that $\mu(A_n) \to 0$. Show that $\lim_{n\to\infty} \int_{A_n} f d\mu = 0$.

Proof. First suppose $f \geq 0$. Let $E_n = f^{-1}([n, \infty])$. Then $n\mu(E_n) \leq \int_{E_n} f \leq \int f < \infty$ implies $\mu(E_n) \to 0$, and furthermore we have $E_n \to E = f^{-1}(\{\infty\})$. For any $x \notin E$,$f(x) < \infty$ so there is $N > f(x)$, so that $x \notin E_n$ for all $n \geq N$. Thus the functions $f_n = f\chi_{E_n}$ converge pointwise to 0 almost everywhere. Further more, $|f\chi_{E_n}| \leq f$ for all n, so the sequence f_n is dominated by the integrable function f. It follows that $\int_{E_n} f \to 0$.

Write $\int_{A_n} f = \int_{A_n \cap E_m} f + \int_{A_n \cap E_m^c} f$. We have $0 \leq \int_{A_n \cap E_m} f \leq \int_{E_m} f$ since $f \geq 0$. On E_m^c, $f \leq m$ by definition. Thus $0 \leq \int_{A_n \cap E_m^c} f \leq m\mu(A_n)$. For $\varepsilon > 0$, choose m large enough so that $\int_{E_m} f < \varepsilon/2$; we can do this since $\int_{E_n} f \to 0$. Then, choose N large enough so that $\mu(A_n) < \varepsilon/(2m)$ for all $n \geq N$; we can do this since $\mu(A_n) \to 0$. Then we have $\int_{A_n} f < \varepsilon/2 + m\varepsilon/(2m) = \varepsilon$ for all $n \geq N$, so $\int_{A_n} f \to 0$ for $f \geq 0$.

In general, we have $f = f_+ - f_-$, where $f_+, f_- \geq 0$. Then $\int_{A_n} f = \int_{A_n} f_+ - \int_{A_n} f_- \to 0 - 0 = 0$. □

b) Prove or disprove: Let $(X_k)_k$ be a sequence in $\mathcal{A}$ such that $\mu(X_k) < \infty$ and $\bigcup X_k = X$. Let $F_n = \bigcup_{k\geq n} X_k$. Then $\lim_{n\to\infty} \int_{F_n} f d\mu = 0$.

Proof. The statement is false. Let $X = \mathbb{R}$ with the Lebesgue measure. Consider $X_k = [-k, k]$. Then $F_n = \bigcup_{k\geq n} X_k = \mathbb{R}$ for all n, so $\lim_{n\to\infty} \int_{F_n} f = \int f$ is not 0 for any function with non-zero integral (say $f(x) = e^{-x^2}$). □

4 Problem 4

Let V be a Banach space with dual V^*. For $u \in V$ define $T(u) : V^* \to \mathbb{F}$ by $T(u)(\varphi) = \varphi(u)$.

a) Show that $T(u) \in (V^*)^*$ and find the norm of $T(u)$.

Proof. Let $\varphi, \eta \in V^*$ and $a \in \mathcal{F}$. Then $T(u)(a\varphi + \eta) = (a\varphi + \eta)(u) = a\varphi(u) + \eta(u) = aT(u)(\varphi) + T(u)(\eta)$, so $T(u)$ is linear.

For any $\varphi \in V^*$, $|T(u)(\varphi)| = |\varphi(u)| \leq ||\varphi||||u||$, so $||T(u)|| \leq ||u||$. Thus $T(u)$ is bounded, so $T(u) \in (V^*)^*$.

Now, let φ be the identity map on V. Then $|T(u)(\varphi)| = |\varphi(u)| = ||u||$. Since $||\varphi|| = 1$, we have that $||u|| \in \{|T(u)(\varphi)| \mid ||\varphi|| = 1\}$. Hence $||T(u)|| \geq ||u||$, since $||T(u)||$ is the least upper bound of this set. From the preceding paragraph, we then have $||T(u)|| = ||u||$. □

b) Show that T is linear, injective, and bounded with $||T|| \leq 1$.

Proof. Let $a \in \mathcal{F}$, $u, v \in V$. Then for any $\varphi \in V^*$, $T(au + v)(\varphi) = \varphi(au + v) = a\varphi(u) + \varphi(v) = aT(u)(\varphi) + T(v)(\varphi) = (aT(u) + T(v))(\varphi)$, so $T(au + v) = aT(u) + T(v)$. Thus T is linear.

Now suppose $u \in \ker T$. That is, for any $\varphi \in V^*$, $T(u)(\varphi) = \varphi(u) = 0$. Then we may take φ to be the identity map, in which case $\varphi(u) = 0$ implies $u = 0$. Thus T is injective.

From part a, $||T(u)|| = ||u||$ for all u, so $||T|| \leq 1$. (In fact $||T|| = 1$.)

□