# MATH 7311 Homework 6

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## 1 Problem 1

For $f \in \mathcal{M}_+$, let $\mu_f$ be the measure $\mu_f(A) = \int_A f d\mu$. Show that for all $g \in \mathcal{M}_+$ we have $\int g d\mu_f = \int f g d\mu$.

*Proof.* First we consider when $g$ is a simple function: $g = \sum_j c_j \chi_{E_j}$ for some partition $E_j$ of $X$. Then $\int g d\mu_f = \sum_j c_j \mu_f(E_j) = \sum_j c_j \int_{E_j} f d\mu = \sum_j c_j \int f \chi_{E_j} d\mu = \int f(\sum_j c_j \chi_{E_j}) d\mu = \int f g d\mu$.

Now, $\int g d\mu_f = \sup\{\int \varphi d\mu_f \mid 0 \leq \varphi \leq g, \varphi \text{ simple}\} = \sup\{\int f \varphi d\mu \mid 0 \leq \varphi \leq g, \varphi \text{ simple}\}$. If $\{\varphi_n\}_n$ is a sequence of simple functions which monotone converges to $g$, then $f\varphi_n \to fg$ monotone as well, since $f$ is non-negative. By the monotone convergence theorem, $\int f\varphi_n d\mu \to \int fg d\mu$. Furthermore, this convergence is monotone increasing. We also have $\int f\varphi d\mu \leq \int fg d\mu$ for all $0 \leq \varphi \leq g$, so that $\int fg d\mu$ is an upper bound for $\{\int f\varphi d\mu \mid 0 \leq \varphi \leq g, \varphi \text{ simple}\}$. Thus $\sup\{\int f\varphi d\mu \mid 0 \leq \varphi \leq g, \varphi \text{ simple}\} = \int fg d\mu$, so $\int g d\mu_f = \int fg d\mu$. □

## 2 Problem 2

Let $f \in \mathcal{M}_+$ and assume that $\int f d\mu < \infty$. Let $\varepsilon > 0$. Show that there exists $A \in \mathcal{A}$ such that $\mu(A) < \infty$ and $\int_A f d\mu > \int f d\mu - \varepsilon$.

*Proof.* Let $E_n = f^{-1}([-n, n])$. Then the $E_n$ are an increasing sequence with limit $X$. For all $n$, $f\chi_{E_n} \leq f$, so $\int f\chi_{E_n} d\mu \leq \int f d\mu$. The integral on the left is also equal to $\int_{E_n} f d\mu$. Since $E_n \subset E_{n+1}$, we have $f\chi_{E_n} \leq f\chi_{E_{n+1}}$. Thus, by the MCT, $\lim_n \int f\chi_{E_n} d\mu \to \int f d\mu$. Since the sequence of integrals is bounded above by $\int f d\mu$, there is some $N$ such that for all $n > N$, $\int f\chi_{E_n} d\mu > \int f d\mu - \varepsilon$. Thus $E_n$ is a set with finite measure satisfying $\int_{E_n} f d\mu = \int f\chi_{E_n} d\mu > \int f d\mu - \varepsilon$. $\square$

## 3 Problem 3

Let $f : [0,1] \to [0,\infty)$ be continuous on $(0,1]$. Assume further that $\lim_{x\to 0^+} f(x) = \infty$ and that the improper Riemann integral $\int_0^1 f(x)dx$ exists. Show that $f$ is Lebesgue integrable on $[0,1]$ and that $\int_{[0,1]} f d\lambda = \int_0^1 f(x)dx$.

*Proof.* For a simple function $\varphi$ with $0 \le \varphi \le f$, we have $\int_{[0,1]} \varphi d\lambda = \int_0^1 \varphi dx \le \int_0^1 f dx$. Thus $\int_0^1 f dx$ is an upper bound for $\{\int \varphi d\lambda \mid 0 \le \varphi \le f, \varphi \text{ simple}\}$, so $f$ is Lebesgue integrable.

Let $n$ be a positive integer. Let $x_j = j/n$ for $j = 0, 1, ..., n$. Let $m_j = \inf_{x\in[x_{j-1},x_j]} f(x)$. Then $\int_0^1 f dx = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n m_j$. Let $\varphi_n = \sum_{j=1}^n m_j \chi_{[x_{j-1},x_j]}$. This is a sequence of simple functions which increases monotone to $f$. Thus $\lim_{n\to\infty} \int_{[0,1]} \varphi_n d\lambda = \int_{[0,1]} f d\lambda$. But $\int_{[0,1]} \varphi_n d\lambda = \sum_{j=1}^n m_j(x_j - x_{j-1}) = \frac{1}{n} \sum_{j=1}^n m_j$, so we have $\int_0^1 f dx = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^n m_j = \lim_{n\to\infty} \int_{[0,1]} \varphi_n d\lambda = \int_{[0,1]} f d\lambda$. □

# 4 Problem 4

For each part, check whether the limit exists, and if so, find the value.

a) $\lim_{n\to\infty} \int_1^n (1 - \frac{x}{n})^n dx$.

*Proof.* We can compute $\int_1^n (1-\frac{x}{n})^n dx$ by taking $u = 1-\frac{x}{n}$, giving $-n \int_{1-\frac{1}{n}}^0 u^n du = \frac{-n}{n+1}(0 - (1 - \frac{1}{n})^{n+1})$, which converges to $e^{-1}$. □

b) $\lim_{n\to\infty} \int_1^{2n} (1 - \frac{x}{n})^n dx$.

*Proof.* Computing the integral, we get $\frac{-n}{n+1}((-1)^{n+1} - (1-\frac{1}{n})^{n+1})$, which doesn't converge, since it tends to alternate between $e^{-1} - 1$ and $e^{-1} + 1$. □