MATH 7311 Homework 5

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1 Problem 1

Let $f : X \to \overline{\mathbb{R}}$ be a measurable function such that $\mu(\{x \in X \mid |f(x)| < \infty\}) > 0$. Show that there exists a measurable set E such that $\mu(E) > 0$ and f is bounded on E.

Proof. Note that $\{x \in X \mid |f(x)| < \infty\} = f^{-1}(\mathbb{R})$. Similarly, we can study $E_n = f^{-1}((-n, n)) = \{x \in X \mid |f(x)| < n\}$ for $n \in \mathbb{N}$, so f is bounded on each E_n. Since f is measurable and $(-n, n)$ is measurable, E_n is measurable. The sets $(-n, n)$ are an increasing sequence with limit $\mathbb{R}$. Then the sets E_n are an increasing sequence with limit $f^{-1}(\mathbb{R})$. Suppose that $\mu(E_n) = 0$ for all n. By continuity from below, $\mu(f^{-1}(\mathbb{R})) = \lim_{n\to\infty} \mu(E_n) = \lim_n 0 = 0$, a contradiction. Thus, for some $N \in \mathbb{N}$, $\mu(E_N) > 0$. $\square$

2 Problem 2

Suppose X is a finite measure space. Let $f : X \to \overline{\mathbb{R}}$ be a measurable function such that f is finite almost everywhere.

a) Show that for every $\varepsilon > 0$ there exists $E \in \mathcal{A}$ such that $\mu(E^c) < \varepsilon$ and f is bounded on E.

Proof. Let N be a null set such that f is finite on $X - N$. In particular, $X - N \subset f^{-1}(\mathbb{R})$. Since X is a finite measure space, $\mu(X - N) = \mu(X) - \mu(N) = \mu(X)$. It follows then that $\mu(f^{-1}(\mathbb{R})) = \mu(X)$. Consider the sets $E_n = f^{-1}((-n, n))$ for $n \in \mathbb{N}$. Since the E_n are an increasing sequence with limit $f^{-1}(\mathbb{R})$, $\mu(X) = \mu(f^{-1}(\mathbb{R})) = \lim_{n\to\infty} \mu(E_n)$. Then for $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $|\mu(X) - \mu(E_N)| < \varepsilon$. In particular, since X is a finite space and $\mu(E_N) \leq \mu(X)$, we have $\mu(E_N^c) < \varepsilon$. Furthermore, f is bounded on E_N by definition: $|f(x)| < N$ for all $x \in E_N$, so we are done. □

b) Give an example where there is no $E \in \mathcal{A}$ such that $\mu(E^c) = 0$ and f is bounded on E.

Proof. Let $X = (0, 1)$ with the Lebesgue measure, $f(x) = \frac{1}{x}$, and $E_n = [\frac{1}{n}, 1) = \{x \in (0, 1) \mid 1 < f(x) \leq n\} = f^{-1}([-n, n])$ for each $n \in \mathbb{N}$. By construction, each E_n is measurable. f being bounded on a set implies that set is contained in some E_N, by taking $N \in \mathbb{N}$ larger than the bound on f. So, if f is bounded on a measurable set E, let N be such that $E \subset E_N$. Then $(0, \frac{1}{N}) = E_N^c \subset E^c$, so $\lambda(E^c) \geq \lambda(E_N^c) = \frac{1}{N}$. Thus $\lambda(E^c) \neq 0$. □

c) Give an example where the statement is false when $\mu(X) = \infty$.

Proof. Let $X = \mathbb{N}$ with the usual σ-algebra $\mathcal{P}(\mathbb{N})$ and the counting measure. Then let $f(x) = x$, $\varepsilon = 1$. If f is bounded on E and $\mu(E^c) < \varepsilon = 1$, then since μ takes values in non-negative integers, we must have $\mu(E^c) = 0$, and therefore $E^c = \emptyset$ and $E = \mathbb{N}$. But f is not bounded on $\mathbb{N}$, a contradiction. □

3 Problem 3

Suppose X is countable with the σ-algebra $\mathcal{P}(X)$ and the counting measure. Show that a sequence f_n of measurable functions converges to f in measure iff $f_n \to f$ uniformly.

Proof. ($\leftarrow$) It true always that uniform converge implies convergence in measure.

($\rightarrow$) Suppose $f_n \to f$ in measure. Let $\varepsilon_0 > 0$, and $\varepsilon_1 = 1$. Then for some $N \in \mathbb{N}$ and all $n \geq N$, $\mu(\{x \in X \mid |f(x) - f_n(x)| \geq \varepsilon_0\}) < \varepsilon_1 = 1$. Since μ takes values of non-negative integers, we must have $\mu(\{x \in X \mid |f(x) - f_n(x)| \geq \varepsilon_0\}) = 0$ for all $n \geq N$. But measure 0 in counting measure implies empty, so it is the case that $\{x \in X \mid |f(x) - f_n(x)| < \varepsilon_0\} = X$ for all $n \geq N$. This shows that $f_n \to f$ uniformly. $\square$