MATH 7311 Homework 4

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1 Problem 1

For $n \in \mathbb{N}$, let $f_n(x) = \sin(\pi x/n)$ for $x \geq 0$. Show that $f_n \to 0$ pointwise, but not uniformly.

Proof. For any $x \geq 0$, πx is finite. Thus $\pi x/n \to 0$, so $f_n(x) \to \sin(0) = 0$.

Suppose $f_n \to 0$ uniformly. Then let $N \in \mathbb{N}$ such that for all $n \geq N$, $|f_n(x) - 0| = |f_n(x)| < 1$ for all $x \geq 0$. However, for any n, $x = n/2$ satisfies $f_n(x) = 1$, a contradiction. Thus f_n does not converge uniformly. $\square$

2 Problem 2

For $p > 0$, define $f_n = n^{-p}\chi_{[n,\infty)}$. Show that $f_n \to 0$ uniformly and in measure.

Proof. Since uniform convergence implies almost uniform convergence, which then implies convergence in measure, it suffices to show that $f_n \to 0$ uniformly.

Let $\varepsilon > 0$. Choose $N > \varepsilon^{-1/p}$, so that $N^{-p} < \varepsilon$. Furthermore, for any $n \geq N$, $n^{-p} \leq N^{-p} < \varepsilon$. For $x < n$, $f_n(x) = 0$, and for $x \geq n$, $f_n(x) = n^{-p} < \varepsilon$, so that for all x, $|f_n(x) - 0| < \varepsilon$. Thus $f_n \to 0$ uniformly. $\square$

3 Problem 3

For $p > 0$, let $f_n = n^p \chi_{[n,\infty)}$. Show that $f_n \to 0$ pointwise, but not almost uniformly and not in measure.

Proof. For $x \geq 0$, there is $N \in \mathbb{N}$ such that $x < N$, i.e. $x \notin [N, \infty)$. Furthermore, for all $n \geq N$, $x \notin [n, \infty)$. Then $f_n(x) = 0$ for all $n \geq N$, so $f_n(x) \to 0$.

Since almost uniformly convergence implies convergence in measure, it suffices to show that f_n does not converge to 0 in measure.

Let $E_n = \{x \in \mathbb{R} \mid |f_n(x) - 0| \geq 1\}$. If $x < n$, then $f_n(x) = 0$, so $x \notin E_n$. If $x \geq n$, then $f_n(x) = n^p \geq 1$, so $x \in E_n$. Therefore, $E_n = [n, \infty)$, so for all n, $\mu(E_n) = \infty$. Then $\mu(E_n) \not\to 0$, so f_n does not converge in measure. $\square$

4 Problem 4

Assume that $p \geq 0$. Define $f_n = n^p \chi_{[n,n+1)}$. Show that $f_n \to 0$ pointwise, but f_n does not converge in measure.

Proof. For any x there is $N \in \mathbb{N}$ such that for all $n \geq N$, $x \notin [n, n+1)$. Then $f_n(x) = 0$ for all $n \geq N$, so $f_n(x) \to 0$.

Let $E_n = \{x \in \mathbb{R} \mid |f_n(x) - 0| \geq 1\}$. If $x < n$, then $f_n(x) = 0$, so $x \notin E_n$. If $x \in [n, n+1)$, then $f_n(x) = n^p \geq 1$, so $x \in E_n$. If $x \geq n+1$, then $f_n(x) = 0$, so $x \notin E_n$. Therefore, $E_n = [n, n+1)$, so for all n, $\mu(E_n) = 1$. Then $\mu(E_n) \not\to 0$, so f_n does not converge in measure. $\square$

5 Problem 5

Let $(X, \mathcal{A}, \mu)$ be a measure space. Let $f_n, g_n : X \to \mathbb{R}$ be sequences of measurable functions, and let $f, g : X \to \mathbb{R}$ be measurable. Assume that $f_n \to f$ almost everywhere and $g_n \to g$ almost everywhere. Show that for every $\alpha \in \mathbb{R}$, we have $\alpha f_n + g_n \to \alpha f + g$ almost everywhere.

Proof. Let N_1 be a null set such that $f_n \to f$ on N_1^C, and let N_2 be a null set such that $g_n \to g$ on N_2^C. Then $N_1 \cup N_2$ is also a null set, since $\mu(N_1 \cup N_2) \leq \mu(N_1) + \mu(N_2) = 0 + 0 = 0$. Furthermore, $(N_1 \cup N_2)^C = N_1^C \cap N_2^C$, so on this set $f_n \to f$ and $g_n \to g$. Then $|\alpha f(x) + g(x) - (\alpha f_n(x) + g_n(x))| \leq |\alpha||f(x) - f_n(x)| + |g(x) - g_n(x)| \to |\alpha|0 + 0 = 0$ for all $x \in N_1^C \cap N_2^C$, which is the complement of a null set, showing that $\alpha f_n + g_n \to \alpha f + g$ almost everywhere. $\square$