MATH 7311 Homework 3

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September 2021

1 Problem 1

Let $f : X \to \overline{\mathbb{R}}$ and $M > 0$. Let $f_M(x) = f(x)$ for $x \in f^{-1}([-M, M])$, $f_M(x) = M$ for $x \in f^{-1}([M, \infty])$, and $f_M(x) = -M$ for $x \in f^{-1}([-\infty, -M])$.

(a) Show that f_M is measurable.

Proof. Let $E = f^{-1}([-M, M]), F = f^{-1}((M, \infty]), G = f^{-1}([\infty, -M))$. Each of these sets is measurable since f is measurable. Then $f_M = f\chi_E + M\chi_F - M\chi_G$, where χ_S denotes the indicator of S. Each χ function in this expression is measurable since each set is measurable. Sums and products of measurable functions are measurable, so f_M is measurable. □

(b) Show that as $M \to \infty$, $f_M(x) \to f(x)$ pointwise.

Proof. Let $x \in X$. If $f(x) \in [-M, M]$, then for all $M' \geq M$, $f(x) \in [-M', M']$. In particular, $f_M(x) = f(x)$ implies $f_{M'}(x) = f(x)$ for all $M' \geq M$.

Suppose $f(x) < \infty$ take $M = f(x)$, so $f(x) \in [-M, M]$, so for all $M' \geq M$, $f_{M'}(x) = f(x)$. Since $f(x)$ is finite, this shows $f_M(x) \to f(x)$.

Now, if $f(x) = \pm\infty$, then $f_M(x) = \pm M$ for all $M > 0$. But $M \to \infty$ implies $f_M(x) = \pm M \to \pm\infty = f(x)$, as desired. □

2 Problem 2

Let $f : [0,1] \to \mathbb{R}$ be defined by $f(x) = x\sin(\frac{\pi}{x})$ for $x \in (0,1]$ and $f(0) = 0$. Show f is not of bounded variation.

Proof. Note that for $x_n = \frac{2}{2n+1}$, $f(x_n) = (-1)^n x_n$. Then $|f(x_{n+1}) - f(x_n)| = \frac{2}{2n+3} + \frac{2}{2n+1} = \frac{8n+8}{4n^2+8n+3}$. Then $\Sigma_{j=1}^{n}|f(x_{j+1}) - f(x_j)| = \Sigma_{j=1}^{n}\frac{8n+8}{4n^2+8n+3} \to \infty$ as $n \to \infty$, since the terms of the series approach $\frac{2}{n}$, implying that it will grow as the harmonic series. Then taking the partitions $\{0, x_n, ..., x_1, 1\}$ for larger and larger n implies that $f(x)$ is not of bounded variation, since the variation on these partitions tends to ∞. $\square$

3 Problem 3

For $\alpha \in \mathbb{R}$ and $E \subset \mathbb{R}$, let $\alpha E = \{\alpha x \mid x \in \mathbb{E}\}$. Let $\lambda_\alpha(E) = \lambda(\alpha E)$, where λ is the Lebesgue measure on $\mathbb{R}$. Show that $\lambda_\alpha = |\alpha|\lambda$.

Proof. If $E = \dot{\bigcup}_j I_j$ is a disjoint union of intervals, then $\alpha E = \dot{\bigcup}_j \alpha I_j$ is a disjoint union of the intervals αI_j. Thus $\lambda(\alpha E) = \Sigma_j \lambda(\alpha I_j)$. It suffices to show that $\lambda_\alpha = |\alpha|\lambda$ on intervals.

If $\alpha = 0$, then $\alpha E = \{0\}$ so $\lambda_\alpha(E) = \lambda(\{0\}) = 0 = |\alpha|\lambda(E)$.

If $\alpha > 0$, then $\alpha(a, b] = (\alpha a, \alpha b]$, so $\lambda_\alpha((a, b]) = \lambda((\alpha a, \alpha b]) = \alpha b - \alpha a = \alpha(b - a) = |\alpha|\lambda((a, b])$.

If $\alpha < 0$, then $\alpha(a, b] = [\alpha b, \alpha a)$, so $\lambda_\alpha((a, b]) = \lambda([\alpha b, \alpha a)) = \alpha a - \alpha b = -\alpha(b - a) = |\alpha|\lambda((a, b])$. □