MATH 7311 Homework 2

Andrea Bourque

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1 Problem 1

Let χ_A be the indicator function for a measurable set A.

1. Show $\chi_{A\cup B} = \chi_A + \chi_B - \chi_{A\cap B}$.

Proof. There are four cases:
(a) $x \in A, x \in B$. Then $x \in A \cap B, x \in A \cup B$, so we have $1 = 1 + 1 - 1$.
(b) $x \in A, x \notin B$. Then $x \notin A \cap B, x \in A \cup B$, so we have $1 = 1 + 0 - 0$.
(c) $x \notin A, x \in B$. Then $x \notin A \cap B, x \in A \cup B$, so we have $1 = 0 + 1 - 0$.
(d) $x \notin A, x \notin B$. Then $x \notin A \cap B, x \notin A \cup B$, so we have $0 = 0 + 0 - 0$. □

2. If $\{E_j\}$ is a decreasing sequence of sets with $E = \bigcap E_j$, then show $\chi_E = \lim_j \chi_{E_j}$.

Proof. If $x \in E$, then $x \in E_j$ for each j. Then $\chi_E(x) = 1$ and for each j, $\chi_{E_j}(x) = 1$, so $\lim_j \chi_{E_j}(x) = 1 = \chi_E(x)$ for $x \in E$. If $x \notin E$, then since the E_j are decreasing, there is some N such that $x \notin E_j$ for all $j > N$. Then $\chi_{E_j}(x) = 0$ for all $j > N$. Thus $\lim_j \chi_{E_j}(x) = 0 = \chi_E(x)$ for $x \notin E$. □

3. If $\{E_j\}$ is an increasing sequence of sets with $E = \bigcup E_j$, then show $\chi_E = \lim_j \chi_{E_j}$.

Proof. If $x \in E$, then since E_j is increasing, there is some N such that $x \in E_j$ for all $j > N$, so $\chi_{E_j}(x) = 1$ for all $j > N$. Thus $\lim_j \chi_{E_j}(x) = 1 = \chi_E(x)$ for $x \in E$. If $x \notin E$, then $x \notin E_j$ for all j, so $\chi_{E_j}(x) = 0$ for all j. Thus $\lim_j \chi_{E_j}(x) = 0 = \chi_E(x)$ for $x \notin E$. □

4. Give an example of a sequence $\{E_j\}$ with $E = \bigcup E_j$ where $\chi_E \neq \lim_j \chi_{E_j}$.

Proof. Consider the alternating sequence $E_{2j} = \emptyset, E_{2j+1} = X$, so $E = X$. Then $\chi_E = 1$ (with 1 denoting the function which is constant and equal to 1), while χ_{E_j} alternates between 0 and 1 (again, 0 and 1 denoting constant functions), and so the sequence does not have a limit. □

2 Problem 2

Let $f : X \to \mathbb{R}$ be so that $\#f(X)$ is finite or countably infinite; $f(X) = \{a_j | j \in J \subset \mathbb{N}\}$. Show that f is measurable iff each of the sets $E_n = \{x \in X | f(x) = a_n\}$ is measurable.

Proof. ($\rightarrow$) Notice that by definition, $E_n = f^{-1}(\{a_n\})$. Since a singleton set is measurable in $\mathbb{R}$ and f is measurable, it follows that each E_n is measurable.

($\leftarrow$) Consider a measurable set $A \subset \mathbb{R}$. Note that $f^{-1}(A) = f^{-1}(A \cap f(X))$, since any element $y \notin f(X)$ has no preimage in X. $A \cap f(X)$ is then a union (at most countable) of some $\{a_j\}$'s, so that $f^{-1}(A \cap f(X))$ is a union (at most countable) of some E_j's. Since each E_j is measurable, their union is measurable, i.e. $f^{-1}(A)$ is measurable. Thus f is measurable. $\square$

3 Problem 3

Let $\{f_n\}$ be a sequence in $\mathcal{M}(X)$. Show that $E = \{x \in X | \lim_n f_n(x) \text{ exists } \}$ is measurable.

Proof. Whenever the limit exists, the lim sup and lim inf exist and are equal. Thus consider $g = \overline{\lim} f_n - \underline{\lim} f_n$. Since the lim sup and lim inf of a sequence of measurable functions are measurable functions, and $\mathcal{M}(X)$ is a vector space, g is measurable. Then $g^{-1}(\{0\})$ is measurable. Note that this set includes the values of x where $f_n(x)$ diverges to $\pm\infty$, but since the singletons containing $\pm\infty$ are measurable, we may exclude them and still get a measurable set, showing that E is measurable. □

4 Problem 4

Let $f : X \to \mathbb{C}$. Show that f is measurable iff $\mathrm{Re}f$ and $\mathrm{Im}f$ are measurable.

Proof. ($\rightarrow$) We know the complex conjugate $\overline{f}$ is measurable. Since $\mathrm{Re}f = \frac{1}{2}(f + \overline{f})$ and $\mathrm{Im}f = \frac{1}{2i}(f - \overline{f})$ are linear combinations of measurable functions, they are also measurable.

($\leftarrow$) Since $f = \mathrm{Re}f + i\ \mathrm{Im}f$ is a linear combination of measurable functions, it is measurable. $\square$