MATH 7290 Homework 5

Andrea Bourque

December 15, 2023

Problem 1

Let $\mathfrak{g} = \mathfrak{sl}_2$ and let $\mathbb{W} = \{e, (12)\}$ be its Weyl group.

(a) Let $Z = St_{\mathcal{N}}$ be the Steinberg variety of $\mathfrak{g}$. Let $\mathcal{B}$ denote the flag variety of $\mathfrak{g}$. For $w \in \mathbb{W}$, Y_w denote the diagonal SL_2 orbit in $\mathcal{B}^2$ consisting of pairs of flags in relative position w. Consider the projection $\pi^2 : Z \to \mathcal{B}^2$. Calculate $(\pi^2)^{-1}(Y_w)$ for each $w \in \mathbb{W}$.

(b) Recall the objects $[\Lambda_w^0], T_w^*$. Show that

$$[\Lambda_e^0] = T_e^*,$$
$$[\Lambda_{(12)}^0] = T_e^* + T_{(12)}^*.$$

Proof. (a) Note that Y_e is just the diagonal in $\mathcal{B}^2$, since all Borels are conjugate. It follows that $Y_{(12)}$ is everything else, i.e. pairs $(\mathfrak{b}_1, \mathfrak{b}_2)$ with $\mathfrak{b}_1 \neq \mathfrak{b}_2$. Then

$$(\pi^2)^{-1}(Y_e) = \{(n, \mathfrak{b}, \mathfrak{b}) \in Z\} \cong \tilde{\mathcal{N}},$$
$$(\pi^2)^{-1}(Y_{(12)}) = \{(n, \mathfrak{b}_1, \mathfrak{b}_2) \in Z | \mathfrak{b}_1 \neq \mathfrak{b}_2\}.$$

(b) From the general theory shown in class, the first equation is true, since e is the minimal element in $\mathbb{W}$. Similarly, we know from general theory that $[\Lambda_{(12)}^0] = nT_e^* + T_{(12)}^*$ for some n. Therefore, we only need to show that $n = 1$. Since $[\Lambda_w^0] * [\Lambda_{w'}^0] = [\Lambda_{ww'}^0]$, it is equivalent to show that $T_{(12)}^* * T_{(12)}^* = -2T_{(12)}^*$. But I'm not sure how to do this. □

Problem 2

Let $\mathfrak{g}$ be a complex semisimple Lie algebra and let $\mathcal{N}$ be the nilpotent cone of $\mathfrak{g}$. Show that for evey $x \in \mathcal{N}$, the trivial representation of the component group $C(x)$ occurs in the $C(x)$-representation $H(\mathcal{B}_x)$.

Proof. The statement means that $\mathrm{Hom}_{C(x)}(\mathbb{C}, H(\mathcal{B}_x))$ is non-zero; there is a non-zero $C(x)$ equivariant map $\mathbb{C} \to H(\mathcal{B}_x)$. It is equivalent to give a non-zero element of $H(\mathcal{B}_x)$ which is fixed by $C(x)$ action. Since the group action permutes the (classes of the) irreducible components, the element given by the sum of fundamental classes of irreducible components is fixed by $C(x)$. It is non-zero since the fundamental classes of irreducible components form a basis for $H(\mathcal{B}_x)$. □

Problem 3

(a) Let $G = SL_3$. Use convolution to show that $H(\mathcal{B}_e)$ is isomorphic to the trivial representation of S_3, where e is any regular nilpotent element in $\mathfrak{sl}_3$.

(b) Verify Springer-Spaltenstein's theorem for $e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

Proof. (a) Since $\mathcal{B}_e$ is a point (regular nilpotents are contained in a unique Borel), we know $H(\mathcal{B}_e)$ is a 1-dimensional space, spanned by the fundamental class of a point: $[\bullet]$. To show $H(\mathcal{B}_e)$ is the trivial representation, it suffices to show $[\Lambda_w^0] * [\bullet] = [\bullet]$ for all $w \in S_3$, or even just for one transposition. But I'm not sure how to do this.

(b) Recall (Yun's notes, 1.3.4) that $\mathcal{B}_e$ is a wedge sum of two copies of $\mathbb{P}^1$; one copy corresponds to flags with $V_2 = \{v_1, v_2\}$, the other copy corresponds to flags with $V_1 = \{v_1\}$, and they intersect at the standard flag. Thus $\mathcal{B}_e$ has two irreducible component: they are the two copies of $\mathbb{P}^1$.

The partition for e is $\lambda = 2 + 1$. Then there are two standard Young tableaux: $\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array}$ and $\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$. The first one corresponds to flags where e acts non-trivially on V/V_1, and the other corresponds to flags where e acts trivially on V_3/V_1. Let $V_\bullet \in \mathcal{B}_e$, and let $V_1 = \langle u_1 \rangle$. Then there is some $u_2 \in V$ such that $\langle u_1, u_2 \rangle = \langle v_1, v_2 \rangle$. Then $V = \langle u_1, u_2, v_3 \rangle$, so that $V/V_1 = \langle [u_2], [v_3] \rangle$. We have $e[u_2] = [eu_2] = [0]$, since $u_2 \in \langle v_1, v_2 \rangle = \ker e$. We also have $e[v_3] = [ev_3] = [v_1]$. Then we see that e acts trivially on V/V_1 if and only if $v_1 \in V_1$, i.e. $V_1 = \langle v_1 \rangle$. The inverse image of the first standard Young tableau is then one of the copies of $\mathbb{P}^1$ with a point removed, and the inverse image of the other tableau is the other copy of $\mathbb{P}^1$. The latter space is already closed and irreducible. The first space has closure equal to the copy of $\mathbb{P}^1$ it is contained in. Thus, the closures of inverse images of standard Young tableau are exactly the irreducible components. $\square$

Problem 4

Let $\hat{S}_n$ denote the set of representatives of the isomorphism classes of irreducible representations of S_n. Use the geometry of Steinberg varieties to show:

(a) $\sum_{\chi \in \hat{S}_n} (\dim(\chi))^2 = |S_n|$.

(b) $\sum_{\chi \in \hat{S}_n} \dim(\chi) = \#\{\text{involutions in } S_n\}$.

Proof. (a) Let $G = SL_n$. Then the Weyl group of G is S_n. Let Z be the Steinberg variety. On one hand, the number of irreducible components of Z is equal to $|S_n|$, since the irreducible components are given by $\overline{T^*_{Y_w}(\mathcal{B}^2)}$ for $w \in S_n$. On the other hand, the irreducible components of Z can be identified as the closures of irreducible components of $Z_{\mathbb{O}} = \tilde{\mathbb{O}} \times_{\mathbb{O}} \tilde{\mathbb{O}}$ for nilpotent orbits $\mathbb{O}$. Furthermore, the irreducible components of $Z_{\mathbb{O}}$ are in correspondence with $C(x)$ orbits on pairs of irreducible components of $\mathcal{B}_x$, for any $x \in \mathbb{O}$. Since G is connected, $C(x)$ is trivial, so we find that the irreducible components of $Z_{\mathbb{O}}$ in this case are in correspondence with pairs of irreducible components of $\mathcal{B}_x$, for any $x \in \mathbb{O}$. Since Springer fibers are equidimensional, $\dim H(\mathcal{B}_x)$ is the number of irreducible components of $\mathcal{B}_x$. From the Springer correspondence, again using the fact that $C(x)$ is trivial, we know that the irreducible representations of S_n are in one to one correspondence with nilpotent orbits $\mathbb{O}$, and in particular, given by $H(\mathcal{B}_x)$ for $x \in \mathbb{O}$. Putting everything together,

$$\sum_{\chi \in \hat{S}_n} (\dim(\chi))^2 = \sum_{\mathbb{O} \subset \mathcal{N}} (\dim(H(\mathcal{B}_x)))^2 = \#\{\text{irreducible components of } Z\} = |S_n|.$$

(b) There is an involution on Z given by swapping the factors of $\tilde{\mathcal{N}}$. Let us analyze which irreducible components of Z are fixed by this involution. On the one hand, an irreducible component may be given by a pair of irreducible components of $\mathcal{B}_x$, for x in some nilpotent orbit $\mathbb{O}$. This irreducible component is then clearly fixed if and only if the two components of $\mathcal{B}_x$ are the same. Then the number of irreducible components of Z fixed by the involution is given by the sum over nilpotent orbits of the number of irreducible components of $\mathcal{B}_x$. As before, we know that the number of irreducible components of $\mathcal{B}_x$ is $\dim H(\mathcal{B}_x)$. We also know that the $H(\mathcal{B}_x)$ as x ranges through representatives for nilpotent orbits gives us the irreducible representations of S_n. In other words, we have

$$\sum_{\chi \in \hat{S}_n} \dim(\chi) = \sum_{\mathbb{O} \subset \mathcal{N}} \dim H(\mathcal{B}_x)$$
$$= \#\{\text{irreducible components of } Z \text{ fixed by swapping factors}\}.$$

On the other hand, the irreducible components of Z are given by closures of conormal bundles to Y_w. This component is fixed by swapping factors if and only if the orbit Y_w is fixed. Explicitly, Y_w is the G orbit $(\mathfrak{b}, w\mathfrak{b})$, so swapping factors gives the G orbit of $(w\mathfrak{b}, \mathfrak{b})$, which is the same as the G orbit of $(\mathfrak{b}, w^{-1}\mathfrak{b})$. Thus, the irreducible component of Z corresponding to w is fixed by swapping factors if and only if $w = w^{-1}$, i.e. w is an involution. Combining this with our previous observation finishes the problem. $\square$

Problem 5

Let R be a commutative ring.

(a) Suppose that Spec(R) is connected, so that R has no nontrivial idempotents. Let $f = a_n t^n + \cdots \in R((t))$. Show that f is invertible in $R((t))$ iff there exists an m such that a_i is nilpotent for $i \leq m$ and a_{m+1} is invertible in R.

(b) Let L be a lattice in $R((t))^n$ and let Λ denote the standard lattice. Show that there exists and $N \in \mathbb{Z}$, $N \geq 0$ such that $t^N \Lambda \subset L \subset t^{-N} \Lambda$.

(c) Let L be a lattice. Show that there exists $f_1, \ldots, f_k \in R$ such that $R = (f_1, \ldots, f_k)$ and $L \otimes_{R[[t]]} R_{f_i}[[t]]$ is a free $R_{f_i}[[t]]$ module for all i.

Proof. (a) We use the following facts from commutative algebra without proof. They are given as exercises in chapter 1 of Atiyah-MacDonald.

Fact 1. Nilpotent elements form an ideal.

Fact 2. A nilpotent plus a unit is a unit.

Fact 3. An element $f = a_0 + a_1 t + \cdots \in R[[t]]$ is a unit iff $a_0 \in R$.

Suppose that $f = a_n t^n + \cdots \in R((t))$ and there exists an m such that a_i is nilpotent for $i \leq m$ and a_{m+1} is invertible in R. Also assume $a_n \neq 0$. Then $m+1 \geq n$. Let $g = a_n t^n + \ldots a_m t^m$. In particular, if $m+1 = n$, then $g = 0$. The elements $a_i t^i$ for $n \leq i \leq m$ are nilpotent, since if $a_i^k = 0$, then $(a_i t^i)^k = a_i^k t^{ik} = 0$. Then g is a finite sum of nilpotent elements, so it is nilpotent by fact 1. Now let $h = a_{m+1} + a_{m+2} t + \ldots$, so that $f - g = t^{m+1} h$. Since a_{m+1} is a unit in R, h is a unit in $R[[t]]$ by fact 3. Since $R[[t]] \subset R((t))$, we have h is a unit in $R((t))$. We then have $t^{-m-1} h^{-1}$ is an inverse to $f - g$. Then $f = g + (f - g)$ is a sum of a nilpotent and a unit, so f is a unit by fact 2.

Conversely, suppose $f = a_n t^n + \cdots \in R((t))$ is a unit, and $a_n \neq 0$. First suppose that a_i is nilpotent for all i. Let $f^{-1} = b_k t^k + \ldots$. Then we have

$$1 = a_n b_{-n} + \cdots + a_{-k} b_k.$$

The right hand side of the equation is a finite linear combination of nilpotent elements in R, so it is nilpotent by fact 1. But 1 cannot be nilpotent (assuming R is not allowed to be the zero ring), so we have a contradiction. Therefore, there is some smallest integer m such that a_{m+1} is not nilpotent, and a_i is nilpotent for $i < m$. In particular, $g = a_n t^n + \cdots + a_m t^m$ is nilpotent, as previously argued. Therefore, we have that $f - g = a_{m+1} t^{m+1} + \ldots$ is a unit by fact 2.

We want to show that a_{m+1} is a unit in R. Since $f - g$ is a unit iff $(f-g)t^{-m-1} = a_{m+1} + a_{m+2} t + \ldots$ is a unit, we rename our element to $f = a_0 + a_1 t + \ldots$, with

a_0 not nilpotent. Let $f^{-1} = b_n t^n + \dots$, with $b_n \neq 0$. If $n \geq 1$, then the lowest possible power of t in ff^{-1} is t, which is impossible. Thus, $n < 1$. If $n = 0$, then upon comparing the coefficients of ff^{-1} and 1, we get $a_0 b_0 = 1$, so a_0 is a unit, as desired. Now suppose $n < 0$. Again, comparing coefficients gives us the following system of equations:

$$\begin{aligned} a_0 b_n &= 0, \\ a_0 b_{n+1} + a_1 b_n &= 0, \\ &\vdots \\ a_0 b_0 + \dots + a_{-n} b_n &= 1. \end{aligned}$$

Now, multiplying the second equation by a_0 gives $a_0^2 b_{n+1}$. Similarly, we get $a_0^{k+1} b_{n+k} = 0$ for $0 \leq k < -n$. Multiplying the last equation by a_0^{-n} gives $a_0^{-n+1} b_0 = a_0^{-n}$. Since a_0 is not nilpotent, the right hand side is not 0. We can multiply this equation by arbitrarily high powers of a_0 to get $a_0^{N+1} b_0 = a_0^N$ for all $N \geq -n$, and the right hand side is never 0. By raising both sides to arbitrarily high powers, we get $a_0^{NM} (a_0 b_0)^M = a_0^{NM}$ for all $N \geq -n$ and $M \geq 1$. The right hand side is still never 0, which shows that $a_0 b_0$ is not nilpotent. Now, consider $x = (a_0 b_0)^{-n}$. Then $x^2 = a_0^{-2n-1} a_0 b_0 b_0^{-2n-1} = a_0^{-2n-1} b_0^{-2n-1} = \dots = a_0^{-n+1} b_0 b_0^{-n} = a_0^{-n} b_0^{-n} = x$. Thus, x is idempotent. By our assumption on R, we either have $x = 0$ or $x = 1$. But we know $a_0 b_0$ is not nilpotent, so $x = 1$. Thus $a_0 a_0^{-n-1} b_0^{-n} = 1$, so a_0 is a unit.

(b) By definition, there are $g_1, \dots, g_m \in R[[t]]$ such that $(g_1, \dots, g_m) = R[[t]]$ and $L_i := L_{g_i} := L \otimes_{R[[t]]} R[[t]]_{g_i}$ is a free $R[[t]]_{g_i}$ module. Choose a basis $v_1^i, \dots, v_n^i$ for each L_i. In terms of the standard basis $e_1^i, \dots, e_n^i$ for Λ_i, write the vectors $v_j^i = f_j^i(1) e_1^i + \dots f_j^i(m) e_m^i$. Further write $f_j^i(k) = g_i^{-a(j,k)} \hat{f}_j^i(k)$, with $\hat{f}_j^i(k) \in R((t))$. In fact, since g_i is a unit in $R[[t]]_{g_i}$, we may rescale our basis to assume $a(j,k) = 0$ for all j and k, so that $f_j^i(k)$ can be represented by an element $\hat{f}_j^i(k) \in R((t))$. Let $-N_i'$ be the minimum of the degrees of $\hat{f}_j^i(k)$ and let N_i'' be the maximum of the degrees of $\hat{f}_j^i(k)$. Then $t^{N_i''} \Lambda_i \subset L_i \subset t^{-N_i'} \Lambda_i$. Let N_i be the maximum of N_i' and N_i'', so $t^{N_i} \Lambda_i \subset L_i \subset t^{-N_i} \Lambda_i$. Finally, let N be the maximum of the N_i. Then $t^N \Lambda_i \subset L_i \subset t^{-N} \Lambda_i$ for all i, so $t^N \Lambda \subset L \subset t^{-N} \Lambda$.

(c) By definition, there are $g_1, \dots, g_m \in R[[t]]$ such that $(g_1, \dots, g_m) = R[[t]]$ and $L_i := L_{g_i} := L \otimes_{R[[t]]} R[[t]]_{g_i}$ is a free $R[[t]]_{g_i}$ module. Order the g_i so that $g_1, \dots, g_k$ have nonzero constant coefficient, and $g_{k+1}, \dots, g_m$ have zero constant coefficient. Since $(g_1, \dots, g_m) = R[[t]]$, there exists elements $a_1, \dots, a_m \in R[[t]]$ such that $a_1 g_1 + \dots a_m g_m = 1$. But the terms $a_{k+1} g_{k+1} + \dots a_m g_m$ can only contribute multiples of t, so $a_1 g_1 + \dots a_k g_k = 1 + tb$, for some $b \in R[[t]]$. If we denote the constant coefficients of a_i and g_i by a_i' and f_i, and we reduce the previous equation mod t, we get $a_1' f_1 + \dots + a_k' f_k = 1$. Thus $(f_1, \dots, f_k) = R$. From here on, we take "for all i" to mean $i \in \{1, \dots, k\}$.

Now, we now that $L \otimes_{R[[t]]} R[[t]]_{g_i}$ is a free $R[[t]]_{g_i}$ module for all i. We want to show that $L \otimes_{R[[t]]} R_{f_i}[[t]]$ is a free $R_{f_i}[[t]]$ module for all i. By Fact 3 stated earlier, g_i is invertible in $R_{f_i}[[t]]$, so the base change to $R_{f_i}[[t]]$ factors through the base change to $R[[t]]_{g_i}$. Since the base change of a free module is a free module, it follows that $L \otimes_{R[[t]]} R_{f_i}[[t]]$ is a free $R_{f_i}[[t]]$ module. $\square$