MATH 7290 Homework 4

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Problem 1

Let X be a non-compact, connected, oriented smooth manifold of dimension n. Show that $H_n(X;\mathbb{C}) = 0$.

Proof. Since X is oriented, we have by Poincaré duality $H_n(X;\mathbb{C}) \cong H_c^0(X;\mathbb{C})$. Since X is non-compact, any compact subset K must be proper. $H_c^0(X;\mathbb{C})$ is the direct limit of $H^0(X, X-K;\mathbb{C})$ as K ranges over compact subsets of X. Since X is connected and $X-K$ is always non-empty, we have $H^0(X, X-K;\mathbb{C}) = 0$ for all K, so $H_c^0(X;\mathbb{C}) = 0$. □

Problem 2

Let X, Y be quasi-projective varieties over $\mathbb{C}$. Show that if $f : X \to Y$ is proper, then it is projective.

Proof. proper = separated, finite type, universally closed. projective = factors into closed immersion $X \to \mathbb{P}^n \times Y$ and projection $\mathbb{P}^n \times Y \to Y$.

Since X is quasi-projective, there is an immersion $i : X \to \mathbb{P}^n$. Since $\mathbb{P}^n$ is separated, the diagonal morphism $\Delta : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{P}^n$ is a closed immersion. Let us show that the following diagram is a pullback square:

$$\begin{array}{ccc} X & \xrightarrow{\quad i \quad} & \mathbb{P}^n \\ {\scriptstyle (1,i)}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times \mathbb{P}^n & \xrightarrow[(i,1)]{} & \mathbb{P}^n \times \mathbb{P}^n \end{array}$$

The diagram obviously commutes. Suppose we are given maps $g : Z \to \mathbb{P}^n$ and $g' : Z \to X \times \mathbb{P}^n$ such that the following diagram also commutes:

$$\begin{array}{ccc} Z & \xrightarrow{\quad g \quad} & \mathbb{P}^n \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle \Delta} \\ X \times \mathbb{P}^n & \xrightarrow[(i,1)]{} & \mathbb{P}^n \times \mathbb{P}^n \end{array}$$

Let $g'_X = \pi_1 g'$ and let $g'_{\mathbb{P}^n} = \pi_2 g'$. Commutativity of this diagram means $ig'_X = g = g'_{\mathbb{P}^n}$. We claim that there is a unique map $g^! : Z \to X$ such that $g' = (1, i)g^!$ and $g = ig^!$. If a map were to exist, the first condition implies that $g'_X = g^!$. Therefore, g'_X is the unique map we are looking for. The other conditions $g'_{\mathbb{P}^n} = ig^!$ and $g = ig^!$ are consistent with this choice by the commutativity relations from before. Therefore, the map $(1, i) : X \to X \times \mathbb{P}^n$ is a closed immersion by Stacks tag 01JU.

Now, since $f : X \to Y$ is proper, the base-change $(f, 1) : X \times \mathbb{P}^n \to Y \times \mathbb{P}^n$ is closed. Thus, the map $(f, i) : X \to Y \times \mathbb{P}^n$ is closed. Also, it is an immersion, since one of its components is an immersion. Since the first component of this map is f, we have shown f is projective.

□

Problem 3

Compute $H_*^{BM}(\mathcal{C}; \mathbb{C})$ of the double cone

$$\mathcal{C} = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\}.$$

Proof. $\mathcal{C}$ can be compactified into $S^2 \vee S^2$ by adding two points to close off each side of the double cone. Thus $H_*^{BM}(\mathcal{C}; \mathbb{C}) = H_*(S^2 \vee S^2, \{*_1, *_2\})$. For simplicity, let $X = S^2 \vee S^2$ and $P = \{*_1, *_2\}$. There is a long exact sequence in reduced homology relating $H_*(X, P)$ in terms of $\tilde{H}_*(X)$ and $\tilde{H}_*(P)$. We have

$$\tilde{H}_i(X) = \begin{cases} \mathbb{C}^2 & i = 2 \\ 0 & i \neq 2, \end{cases}$$

$$\tilde{H}_i(P) = \begin{cases} \mathbb{C} & i = 0 \\ 0 & i \neq 0. \end{cases}$$

For $i > 2$, the long exact sequence reads $0 \to H_i^{BM}(\mathcal{C}) \to 0$, so $H_i^{BM}(\mathcal{C}) = 0$. For $i = 2$, the long exact sequence reads $0 \to \mathbb{C}^2 \to H_2^{BM}(\mathcal{C}) \to 0$, so $H_2^{BM}(\mathcal{C}) = \mathbb{C}^2$. Finally, the tail end of the sequence reads

$$0 \to H_1^{BM}(\mathcal{C}) \to \mathbb{C} \to 0 \to H_0^{BM}(\mathcal{C}) \to 0,$$

so $H_1^{BM}(\mathcal{C}) = \mathbb{C}$ and $H_0^{BM}(\mathcal{C}) = 0$. In summary,

$$H_i^{BM}(\mathcal{C}) = \begin{cases} \mathbb{C}^2 & i = 2 \\ \mathbb{C} & i = 1 \\ 0 & \text{else.} \end{cases}$$

□

Problem 4

Let X be an algebraic variety over $\mathbb{C}$ equipped with a filtration by closed subvarieties

$$X = X_m \supset X_{m-1} \supset ... \supset X_0 = \varnothing$$

such that

$$X_i - X_{i-1} \cong \bigsqcup_{j=1}^{k_i} \mathbb{A}^{n_{i,j}}.$$

Show that the classes $[\overline{\mathbb{A}^{n_{i,j}}}]$ form a $\mathbb{C}$-basis for $H_*^{BM}(X)$.

Proof. (Note: I will not be writing BM). We induct on m. For the base case, X is a finite disjoint union of affine spaces. Let $U \subset X$ be the union of the highest dimensional (say complex dimension d) affine spaces appearing in X, and let Z be the complement of U. We know $H_i(U)$ only lives in $i = 2d$, with basis corresponding to each affine space in U (they are the (top dimensional) irreducible components of U). By induction on d, we can assume that $H_i(Z)$ is only nonzero for even i with $i < 2d$. We have an exact sequence

$$\cdots \to H_{i+1}(U) \to H_i(Z) \to H_i(X) \to H_i(U) \to H_{i-1}(Z) \to \dots$$

By our hypotheses, we get an isomorphism $H_i(X) \xrightarrow{\sim} H_i(U)$ only when $i = 2d$, an isomorphism $H_i(X) = 0$ for $i > 2d$, and an isomorphism $H_i(Z) \xrightarrow{\sim} H_i(X)$ for i even and less than $2d$. This shows that the classes corresponding to the affine spaces give a basis for $H_*(X)$.

Now assume the result is true for spaces with a filtration $X = X_{m-1} \supset \cdots \supset X_0 = \varnothing$, and let X have a filtration $X = X_m \supset \cdots \supset X_0 = \varnothing$. Then the result holds for closed subset $Z := X_{m-1}$ and open subset $U := X - X_{m-1}$, the latter of which is a union of affine spaces. Once again we have an exact sequence

$$\cdots \to H_{i+1}(U) \to H_i(Z) \to H_i(X) \to H_i(U) \to H_{i-1}(Z) \to \dots$$

Now, $H_i(Z) = 0 = H_i(U)$ for i odd, so $H_i(X) = 0$ as well. For i even, $i - 1$ and $i + 1$ are odd, so we have a short exact sequence of vector spaces, implying $H_i(X) \cong H_i(Z) \oplus H_i(U)$. In particular, the bases for $H_*(Z)$ and $H_*(U)$ combine to give us a basis for $H_*(X)$ as desired. □

Problem 5

Use Problem 4 to do these computations.

(1) Compute the singular cohomology of $\mathbb{CP}^n$.

(2) Let G be a complex semisimple algebraic group and let B be a Borel subgroup. Compute the singular cohomology of G/B.

Proof. (1) We have a filtration $X_j = \mathbb{CP}^{j-1}$ for $0 < j \leq n+1$ with $X_j - X_{j-1} = \cong \mathbb{A}^{j-1}$. Each $\mathbb{A}^{j-1}$ is dense in X_j, so $\overline{\mathbb{A}^{j-1}} = X_j$. Thus, the $[X_j]$ form a basis for $H_*^{BM}(\mathbb{CP}^n)$, with $[X_j] \in H_{2(j-1)}^{BM}(\mathbb{CP}^n)$. By Poincaré duality, the singular cohomology will also have 1 generator in each even degree, given by the duals to the classes $[X_j]$.

(2) By the Bialynicki Birula decomposition, and by a previous homework problem (essentially the Bruhat decomposition, but we need the affine part), we know that G/B is a finite disjoint union of affine spaces. Explicitly, the affine spaces are the B orbits of the coset wB for $w \in W_T$ for a maximal torus $T \subset B$. These spaces have dimension given by the length of w. The closure of $B(wB)$ is the union of $B(w'B)$ for $w' \leq w$ in the Bruhat order. By Problem 4, the classes of these closures give us a basis for the Borel Moore homology of G/B. By duality, the duals of these classes give us a basis for the singular cohomology of G/B. $\square$