MATH 7290 Homework 3

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Problem 1

Let $A \in \mathfrak{sl}_n(\mathbb{C})$ be the regular nilpotent matrix defined by $A_{i,j} = 1$ if $j = i+1$ and $A_{i,j} = 0$ otherwise. Show directly that the standard flag is the only complete flag preserved by A.

Proof. Fix a basis $\{e_i\}$ for $\mathbb{C}^n$. Suppose A preserves the flag

$$0 \subset \langle v_1 \rangle \subset \langle v_1, v_2 \rangle \subset \cdots \subset \langle v_1, \dots, v_n \rangle = \mathbb{C}^n.$$

Since A is nilpotent, preserving the flag means $AV_i \subset V_{i-1}$. In particular, $Av_1 = 0$. Since $\ker A = \langle e_1 \rangle$, we have without loss of generality $v_1 = e_1$. Assume by induction (and without loss of generality) that $v_i = e_i$ for $i < j$, for some $j \in \{2, \dots, n-1\}$. We have $Av_j \in \langle e_1, \dots, e_{j-1} \rangle$. In particular, the e_k component of Av_j for $k \in \{j, \dots, n-1\}$ is 0. Since the e_k component of Av_j is the e_{k+1} component of v_j, this implies that $v_j \in \langle e_1, \dots, e_j \rangle$. Since v_j is linearly independent from $e_1, \dots, e_{j-1}$, we can $v_j = e_j$ without loss of generality. Finally, if $v_i = e_i$ for $i < n$, then since v_n is linearly independent from the e_i, we can take it to be e_n without loss of generality. Thus, our flag is the standard flag. $\square$

Problem 2

Consider $G = Sp(V)$, $V = \langle v_1, v_2, v_3, v_4\rangle$ with the symplectic form given by $\langle v_i, v_{5-i}\rangle = 1$ if $i = 1, 2$ and $\langle v_i, v_j\rangle = 0$ for $i + j \neq 5$. Let $e : v_4 \mapsto v_1 \mapsto 0, v_2 \mapsto 0, v_3 \mapsto 0$. Then any flag $0 \subset V_1 \subset V_2 \subset V_1^\perp \subset V$ in $\mathcal{B}_e$ must satisfy $\langle v_1\rangle \subset V_2 \subset \langle v_1, v_2, v_3\rangle$. Moreover, this is the only condition for a flag $0 \subset V_1 \subset V_2 \subset V_1^\perp \subset V$ to lie in $\mathcal{B}_e$.

Proof. From Yun's notes, for a flag $0 \subset V_1 \subset V_2 \subset V_1^\perp \subset V$ to lie in $\mathcal{B}_e$, we must have $V_2 = V_2^\perp$ and $eV_i \subset V_{i-1}$, where $V_0 = 0$, $V_3 = V_1^\perp$, and $V_4 = V$. Therefore, we must show that these conditions are equivalent to $\langle v_1\rangle \subset V_2 \subset \langle v_1, v_2, v_3\rangle$.

Suppose $\langle v_1\rangle \subset V_2 \subset \langle v_1, v_2, v_3\rangle$. Since $\langle v_1, v_2, v_3\rangle = \ker e$, we have $eV_2 = 0 \subset V_1$. If $eV_1 \neq 0$, then V_1 must contain some element with a non-zero v_4 component. But $V_1 \subset V_2 \subset \langle v_1, v_2, v_3\rangle$, and the latter space contains no such vectors, a contradiction. Thus, $eV_1 = 0$. Since $\mathrm{im} e = \langle v_1\rangle$, we have that $eV_3 \subset \langle v_1\rangle \subset V_2$ and $eV_4 \subset \langle v_1\rangle \subset V_2 \subset V_3$. Thus, e preserves the flag.

It remains to show that $V_2 = V_2^\perp$. Since $v_1 \in V_2$ and $V_2 \subset \langle v_1, v_2, v_3\rangle$, we can take $V_1 = \langle v_1, av_2 + bv_3\rangle$ where a, b are constants and not both 0. By direct computation, we can show

$$\langle cv_1 + adv_2 + bdv_3, ev_1 + afv_2 + bfv_3\rangle = abdf - abdf = 0,$$

for all constants c, d, e, f. Thus $V_2 \subset V_2^\perp$. Now let $w \in V_2^\perp$ and write $w = \sum_i w^i v_i$. Since $\langle v_1, w\rangle = w^4$ and $v_1 \in V_2$, we must have $w^4 = 0$. Expanding the relation $\langle w, cv_1 + adv_2 + bdv_3\rangle = 0$ for all c, d, we have $bdw^3 = adw^2$ for all c, d. Taking $d = 1$, we get $bw^3 = aw^2$. Since a, b are not both 0, this implies that $w^2 v_2 + w^3 v_3$ is a multiple of $av_2 + bv_3$. Thus $w \in V_2$, which shows that $V_2^\perp \subset V_2$. Thus, $V_2 = V_2^\perp$. This concludes one direction of the proof.

Now suppose that $eV_i \subset V_{i-1}$ and $V_2 = V_2^\perp$. Since $eV_1 = 0$, we have $V_1 \subset \langle v_1, v_2, v_3\rangle$. If $V_2 \not\subset \langle v_1, v_2, v_3\rangle$, say $w \in V_2$ has a non-zero v_4 component w^4, then $w^4 v_1 = ew \in eV_2 \subset V_1$, so $V_1 = \langle v_1\rangle$. But $V_2 \subset V_1^\perp$ implies $w \in V_1^\perp$, and $\langle v_1, w\rangle = w^4 \neq 0$, a contradiction. Thus $V_2 \subset \langle v_1, v_2, v_3\rangle$. Finally, $v_1 \in \langle v_1, v_2, v_3\rangle^\perp \subset V_2^\perp = V_2$, so $\langle v_1\rangle \subset V_2$. Thus, $\langle v_1\rangle \subset V_2 \subset \langle v_1, v_2, v_3\rangle$. $\square$

Problem 3

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Let $x \in \mathfrak{b}$. Show that x is nilpotent in $\mathfrak{g}$ if and only if $x \in \mathfrak{n}$.

Proof. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{b}$, as well as a root system and a base of simple roots. Then $\mathfrak{n} = \bigoplus_{\alpha>0} \mathfrak{g}_\alpha$ and $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$.

To show that $x \in \mathfrak{n}$ is nilpotent, it suffices by linearity to show that for any $\alpha > 0$, $x \in \mathfrak{g}_\alpha$ is nilpotent. This follows from the fact that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$. In particular, $(\mathrm{ad}x)^n : \mathfrak{g}_\beta \to \mathfrak{g}_{\beta+n\alpha}$, and the set of roots is finite, so for large enough n, we will have $(\mathrm{ad}x)^n : \mathfrak{g}_\beta \to 0$ for all β. Since $\mathfrak{g} = \bigoplus_\alpha \mathfrak{g}_\alpha$, we have $(\mathrm{ad}x)^n = 0$, so x is nilpotent.

Now suppose arbitrary $x \in \mathfrak{b}$ is nilpotent. It admits a decomposition $x = h + n$, where $h \in \mathfrak{h}$ and $n \in \mathfrak{n}$. As we have already shown, n is nilpotent. Thus, $h = x - n$ is nilpotent. If $(\mathrm{ad}h)^n = 0$, then for any $y \in \mathfrak{g}_\alpha$, we have $\alpha(h)^n y = 0$. Thus $\alpha(h) = 0$ for all roots α, which means that $h = 0$. Thus $x = n$ is in $\mathfrak{n}$. $\square$

Problem 4

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$. Let $x \in \mathfrak{b}$ be a regular semisimple element. Show that $x + \mathfrak{n}$ is equal to $B \cdot x$. Now deduce that for any G-invariant polynomial P on $\mathfrak{g}$, the restriction of P to $y + \mathfrak{n}$ is constant for any $y \in \mathfrak{b}$.

Proof. Since x is semisimple and regular, its centralizer in $\mathfrak{g}$ is a Cartan subalgebra, which we can identify with the Lie algebra of a maximal torus T. Then write $B = UT$. We have $B \cdot x = UT \cdot x = U \cdot x$. Furthermore, the stabilizer of U acting on x is exactly $U \cap T = \{\mathrm{id}\}$, so $\dim(U \cdot x) = \dim(U)$. Since $\mathfrak{n}$ is the Lie algebra of U, we have $\dim(U) = \dim(\mathfrak{n}) = \dim(x + \mathfrak{n})$. A general result states orbits under unipotent group actions are closed, so $U \cdot x$ is closed. Finally, it suffices to show that $x + \mathfrak{n}$ is U-stable. U acts trivially on B/U, so it also acts trivially on the Lie algebra of B/U, which is $\mathfrak{b}/\mathfrak{n}$. In particular, it sends $x + \mathfrak{n}$ to $x + \mathfrak{n}$. Thus, $U \cdot x \subset x + \mathfrak{n}$. Since $U \cdot x$ is closed and of the same dimension as $x + \mathfrak{n}$, we have $U \cdot x = x + \mathfrak{n}$.

If $y \in \mathfrak{b}$ is regular and semisimple, then by the first part, $P|_{y+\mathfrak{n}} = P|_{B \cdot y} = P(y)$. If y is not regular semisimple, then since semisimple regular elements are dense, we can approximate y by semisimple regular elements and use continuity of polynomials to conclude $P|_{y+\mathfrak{n}}$ is constant. $\square$

Problem 5

Let X be a variety over $\mathbb{C}$.

a. Recall the following fact from algebraic geometry: the smooth locus X^{sm} is open and dense in X. Now show that X^{sm} intersects each irreducible component of X.

b. Show that X is equidimensional if and only if for every non-empty open subset $U \subset X$, we have $\dim(U) = \dim(X)$.

Proof. a. It is more generally true that any dense subset A of X intersects each irreducible component of X. Let the irreducible components of X be $X_1, \dots, X_m$. They are all closed subsets of X. Then $\overline{A \cap X_i} \subset X_i$, and $X = \overline{A} = \bigcup_{i=1}^{m} \overline{A \cap X_i}$. Suppose some $A \cap X_j = \varnothing$, so that $X_j \subset X = \bigcup_{i \neq j} \overline{A \cap X_i} \subset \bigcup_{i \neq j} X_i$. Then $X_j = \bigcup_{i \neq j} X_i \cap X_j$. Each $X_i \cap X_j$ is a closed subset, and X_j is irreducible, so there is some $i \neq j$ such that $X_j = X_i \cap X_j$. This means $X_j \subset X_i$, which contradicts X_j being an irreducible component.

b. If $\dim(U) = \dim(X)$ for all non-empty opens U, then take $U = X - \bigcup_{i \neq j} X_i$. This is a non-empty open subset of X_j, so it is dense in X_j, so it has dimension $\dim(X_j)$. This shows $\dim(X_j) = \dim(X)$ for all j, so X is equidimensional.

Now suppose X is equidimensional. A general fact about dimension is that $Y \subseteq Z$ implies $\dim(Y) \leq \dim(Z)$. Thus $\dim(U) \leq \dim(X)$. Now, U must intersect some irreducible component X_i. Then $U \cap X_i$ is a non-empty open subset of the irreducible space X_i, meaning $\dim(U \cap X_i) = \dim(X_i)$. Since X is equidimensional, this gives $\dim(U \cap X_i) = \dim(X)$. Since $U \cap X_i \subseteq U$, we have $\dim(U \cap X_i) \leq \dim(U)$, so $\dim(X) \leq \dim(U)$. Thus, $\dim(U) = \dim(X)$. $\square$