MATH 7290 Homework 2

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1 Problem 1

Find the regular elements in $\mathfrak{sl}_n(\mathbb{C})$ up to conjugacy.

Proof. We claim that the regular elements $X \in \mathfrak{sl}_n(\mathbb{C})$ are exactly those elements whose Jordan Normal Form (JNF) does not contain two blocks with the same eigenvalue.

Since a matrix is determined up to conjugacy by its JNF, we will simply work with the JNFs directly. Suppose X is a direct sum of k Jordan blocks Λ_i with size l_i and eigenvalue λ_i. Given an $n \times n$ matrix A, write it in corresponding $k \times k$ block form, with the $l_i \times l_j$ submatrix of A at block position (i, j) being denoted A_{ij}. Since X is block diagonal, it is clear to see that $AX = XA$ if and only if $A_{ij}\Lambda_j = \Lambda_i A_{ij}$ for all (i, j).

We now drop the i, j and suppose that Λ, Λ' are Jordan blocks of size p and q respectively and with eigenvalues λ and λ' respectively. Also suppose A is a $p \times q$ matrix satisfying $\Lambda A = A\Lambda'$. Write $\Lambda = \lambda I_p + N$ and $\Lambda' = \lambda' I_q + N'$. Explicitly, the N and N' matrices have 0's everywhere except the elements above the diagonal. Then $\Lambda A = \lambda A + NA$ and $A\Lambda' = \lambda' A + AN'$. Therefore, our equation may be written as

$$(\lambda - \lambda')A = NA - AN'. \tag{1}$$

We now analyze what form A has in the two cases where $\lambda = \lambda'$ and $\lambda \neq \lambda'$.

First consider the case where $\lambda = \lambda'$. Then we just have $NA = AN'$. Letting the (i, j) entry of A be $a_{i,j}$, and comparing the (i, j) entries of NA and AN', we get

$$a_{i+1,j} = a_{i,j-1}, \tag{2}$$

where it is understood that if $i = p$ the left hand side vanishes, and if $j = 1$ the right hand side vanishes. It is annoying to describe exactly the form of matrices satisfying this condition, but there are two important observations that suffice for us:

Observation 1. If $p = q$, then A is upper triangular and each diagonal contains the same elements (succinctly, an upper triangular Toeplitz matrix).

Observation 2. Regardless of the relationship of p and q, there is at least one non-zero matrix A satisfying Equation (2), namely the matrix with 0's everywhere but the $(1, q)$ entry (and that entry can be any non-zero number).

We can now identify a dimension $n - 1$ subspace of the centralizer of X. Consider matrices A in $\mathfrak{sl}_n(\mathbb{C})$ written in block form corresponding to the JNF of X. In order for $AX = XA$, it is necessary that the diagonal blocks of A must be upper triangular Toeplitz matrices (by the above case, seeing as $\lambda_i = \lambda_i$). Consider only such A that are block diagonal. Certainly, such an A commutes with X, since all of the equations $\Lambda_i A_{ij} = A_{ij}\Lambda_j$ are true for $i \neq j$ since $A_{ij} = 0$, and for $i = j$ it is true by our work above (observation 1). In each diagonal block of A, we have l_i independent variables. However, in order for A to be in $\mathfrak{sl}_n(\mathbb{C})$, we must have one linear condition on the diagonal elements. Therefore, the set of block diagonal A, where each block on the diagonal is upper triangular Toeplitz, has dimension $n - 1$. As we have already mentioned, each of these matrices commute with X, so this is a subspace of $Z_{\mathfrak{g}}(X)$.

Recall that X is regular if and only if $\dim Z_{\mathfrak{g}}(X) = \operatorname{rank}\mathfrak{g}$. The rank of $\mathfrak{sl}_n(\mathbb{C})$ is $n - 1$. Thus, we see that X is regular if and only if the only elements which commute with X are exactly those described above. By our observation 2, if some $\lambda_i = \lambda_j$ for $i \neq j$, then A_{ij} can be made non-zero, meaning there is some non-block diagonal matrix in $Z_{\mathfrak{g}}(X)$, so that X is not regular. In other words, if X has repeated eigenvalues in its JNF, it cannot be regular.

It remains to prove the converse; if X has distinct eigenvalues in its JNF, then it is regular. As we already showed, the block diagonal matrices whose diagonal blocks are upper triangular Toeplitz comprise an $n - 1$ dimensional subspace of the centralizer of X. Therefore, it suffices to show that if $i \neq j$, $\lambda_i \neq \lambda_j$), and $\Lambda_i A_{ij} = A_{ij}\Lambda_j$, then $A_{ij} = 0$.

Once again, we ditch i, j and return to Equation (1) and its notations. Furthermore, we won't really need λ and λ'; we only need the fact that $\lambda - \lambda' \neq 0$. To that end, set $b = \lambda - \lambda'$. Equating the (i, j) entries on both sides of Equation (1) gives:

$$ba_{i,j} = a_{i+1,j} - a_{i,j-1}. \tag{3}$$

Once again, we understand that if $j = 1$, the term $a_{i,j-1}$ is 0, and if $i = p$, then $a_{i+1,j} = 0$. This gives us two convenient equations:

$$ba_{i,1} = a_{i+1,1}, \tag{4}$$

$$ba_{p,j} = -a_{p,j-1}. \tag{5}$$

In particular, we have $ba_{p,1} = 0$, so $a_{p,1} = 0$. Then Equation (4) gives $ba_{p-1,1} = 0$ and Equation (5) gives $ba_{p,2} = 0$. It is clear to see by induction that the left most column and bottom most row of A must vanish in this way. Then Equations (3), (4), and (5) simply restrict to constraints on the top-right $(p-1) \times (q-1)$ submatrix of A. By induction, we conclude $A = 0$, as desired. This finishes the proof.

$\square$

2 Problem 2

Let G be a complex semisimple algebraic group and let $\mathfrak{g}$ be its Lie algebra. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the Killing form.

a. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Show that the restriction of κ to $\mathfrak{h} \times \mathfrak{h}$ is non-degenerate.

b. Let B be a Borel subgroup of G with Lie algebra $\mathfrak{b}$ and let $\mathfrak{n}$ be its nilradical. Show that under the isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$ induced by the Killing form, $\mathfrak{n}$ gets B-equivariantly identified with $\mathfrak{b}^\perp$.

Proof. a. Given $\mathfrak{h}$, we have a root system and a corresponding decomposition $\mathfrak{g} = \mathfrak{h} \bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$. We can also write $\mathfrak{h} = \mathfrak{g}_0$.

We now show that if $\alpha, \beta \in \Phi \cup \{0\}$ with $\alpha + \beta \neq 0$, then $\kappa(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0$. Let $h \in \mathfrak{h}$ be such that $(\alpha + \beta)(h) \neq 0$. Let $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. We have $\kappa([h, x], y) = -\kappa([x, h], y) = -\kappa(x, [h, y])$, with the second equality due to "associativity" of the Killing form. Since $[h, x] = \alpha(h)x$ and $[h, y] = \beta(h)y$, this equation reads $\alpha(h)\kappa(x, y) = -\beta(h)\kappa(x, y)$, or $(\alpha + \beta)(h)\kappa(x, y) = 0$. Thus $\kappa(x, y) = 0$ as desired.

As a consequence, $\mathfrak{h}$ is κ-orthogonal to every root space. If some $h \in \mathfrak{h}$ satisfied $\kappa(h, \mathfrak{h}) = 0$, then we would have $\kappa(h, \mathfrak{g}) = 0$. But κ is non-degenerate since $\mathfrak{g}$ is semisimple, so $h = 0$. Thus, κ is non-degenerate on $\mathfrak{h} \times \mathfrak{h}$.

b. Let $\mathfrak{h}$ be a Cartan subalgebra contained in $\mathfrak{b}$. Pick a corresponding root system, as well as simple roots so that we have a partition $\Phi = \Phi^- \coprod \Phi^+$. Then we have $\mathfrak{b} = \mathfrak{h} \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ and $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}] = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$. Since $\mathfrak{g} = \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$, $\dim g_\alpha = 1$ for $\alpha \in \Phi$, and $|\Phi^-| = |\Phi^+|$, we have $\dim \mathfrak{n} = |\Phi^+| = |\Phi^-| = \dim \bigoplus_{\alpha \in \Phi^-} \mathfrak{g}_\alpha = \dim \mathfrak{g} - \dim \mathfrak{b}$. Since we also have $\dim \mathfrak{b}^\perp = \dim \mathfrak{g} - \dim \mathfrak{b}$, it follows that $\dim \mathfrak{n} = \dim \mathfrak{b}^\perp$.

Now we show that $\mathfrak{n}$ is mapped to $\mathfrak{b}^\perp$ under $x \mapsto \kappa(x, -)$. By linearity, it suffices to take $x \in \mathfrak{g}_\alpha$ for $\alpha \in \Phi^+$. If $\beta \in \Phi^+ \cup \{0\}$, then $\alpha + \beta \neq 0$, so $\kappa(x, \mathfrak{g}_\beta) = 0$ by the result in part a. Since $\mathfrak{b}$ is the direct sum of such g_β, we have $\kappa(x, \mathfrak{b}) = 0$. Since κ is non-degenerate, this mapping $\mathfrak{n} \to \mathfrak{b}^\perp$ is injective. An injective linear map between vector spaces of the same (finite) dimension is an isomorphism, so $\mathfrak{n}$ and $\mathfrak{b}^\perp$ are identified.

Now we show that this identification is B-equivariant. For $x \in \mathfrak{n}$ and $b \in B$, we have $\mathrm{Ad}(b)x$ is mapped to $\kappa(\mathrm{Ad}(b)x, -)$. Since κ is invariant under automorphisms of $\mathfrak{g}$, we have $\kappa(\mathrm{Ad}(b)x, -) = \kappa(x, -) \circ \mathrm{Ad}(b)^{-1} = \kappa(x, -) \circ \mathrm{Ad}(b^{-1})$. By definition of the coadjoint action of B on $\mathfrak{g}^*$, we have $\kappa(x, -) \circ \mathrm{Ad}(b^{-1}) = \mathrm{Ad}^*(b)\kappa(x, -)$. Thus, the identification is B-equivariant. ☐

3 Problem 3

Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$. Let $X \in \mathfrak{g}$ be a nilpotent element. Show that for any $\lambda \in \mathbb{C}^\times$, λX is nilpotent.

Proof. Note that $\mathrm{ad}(\lambda X) = \lambda \mathrm{ad}(X)$, so if $\mathrm{ad}(x)^n = 0$, then $\mathrm{ad}(\lambda X)^n = 0$. $\square$

4 Problem 4

Let $\mathcal{B}$ denote the flag variety of a semisimple algebraic group G over $\mathbb{C}$. View $\mathcal{B}$ as G/B for a Borel subgroup B of G and let $T \subset B$ be a maximal torus in G. Show that for every regular cocharacter $\lambda : \mathbb{G}_m \to T$, we have $\mathcal{B}^T = \mathcal{B}^{\lambda(\mathbb{G}_m)}$.

Proof. Since $\lambda(\mathbb{G}_m) \subseteq T$, we have $\mathcal{B}^T \subseteq \mathcal{B}^{\lambda(\mathbb{G}_m)}$. Thus, it suffices to show the opposite containment.

First, we show that $\mathcal{B}^{\lambda(\mathbb{G}_m)}$ is T-stable. Suppose $gB \in \mathcal{B}^{\lambda(\mathbb{G}_m)}$, so that $\lambda(a)gB = gB$ for all $a \in \mathbb{G}_m$. Then for $t \in T$, we have $t\lambda(a)gB = tgB$ for all $a \in \mathbb{G}_m$. Since T is abelian, $t\lambda(a) = \lambda(a)t$, so we get $\lambda(a)tgB = tgB$ for all $a \in \mathbb{G}_m$, i.e. $tgB \in \mathcal{B}^{\lambda(\mathbb{G}_m)}$. Thus, $\mathcal{B}^{\lambda(\mathbb{G}_m)}$ is T-stable.

Now we show that $\mathcal{B}^{\lambda(\mathbb{G}_m)}$ is finite. By Bialynicki-Birula decomposition, we know that $\mathcal{B}^{\lambda(\mathbb{G}_m)}$ is a smooth variety, so its connected components are its irreducible components, and there are finitely many of them, since a variety is Noetherian. Call the connected components Z_i. For $z \in Z_i$, we know that $\dim(Z_i) = \dim((T_z\mathcal{B})^0)$, where $(T_z\mathcal{B})^0$ is the subspace of $T_z\mathcal{B}$ where $\mathbb{G}_m$ acts trivially. Identifying $\mathcal{B}$ with G/B for some Borel subgroup B, let $z = gB$. Then we can identify $T_{gB}(\mathcal{B})$ with $\mathfrak{g}/(\mathrm{Ad}(g)\mathfrak{b})$. This quotient will be a sum of some root spaces, since the Borel subalgebra will cancel out a Cartan subalgebra of $\mathfrak{g}$. Since λ is regular, it does not fix any roots. This shows $(T_z(\mathcal{B}))^0 = 0$ for $z \in Z_i$, so $\dim(Z_i) = 0$. An irreducible space of dimension 0 is a point, so $\mathcal{B}^{\lambda(\mathbb{G}_m)}$ is a finite union of points.

Finally, since T is connected, its continuous action on a finite set of disconnected points must be trivial. Thus, every point in $\mathcal{B}^{\lambda(\mathbb{G}_m)}$ is in $\mathcal{B}^T$, as desired. $\square$

5 Problem 5

Let G be a linear algebraic group over $\mathbb{C}$ acting on an affine variety X over $\mathbb{C}$. Let $\mathbb{C}[X]$ denote the ring of regular functions on X. Then G acts on $\mathbb{C}[X]$ in the usual way. Show that every non-zero $f \in \mathbb{C}[X]$ is contained in a finite-dimensional G-invariant subspace of $\mathbb{C}[X]$.

Proof. Corresponding to the action map $\phi : G \times X \to X$, let ϕ^* be the map on coordinate rings $\mathbb{C}[X] \to \mathbb{C}[G \times X]$. It is well known that the product of affine varieties has coordinate ring isomorphic to the tensor product of the constituent coordinate rings. In less words, we have $\mathbb{C}[G \times X] \cong \mathbb{C}[G] \otimes \mathbb{C}[X]$. Thus, we can identify ϕ^* with a map $\phi^* : \mathbb{C}[X] \to \mathbb{C}[G] \otimes \mathbb{C}[X]$.

Now, let $\phi^* f = \sum_{i=1}^n a_i \otimes b_i \in \mathbb{C}[G] \otimes \mathbb{C}[X]$. Explicitly, this means that for $g \in G$, we have $(gf)(x) = f(g^{-1}x) = \sum a_i(g^{-1})b_i(x)$. It follows that $gf = \sum a_i(g^{-1})b_i$ is in the span of the b_i, which is finite dimensional. Then the span of the G-orbit of f is also finite dimensional, and G-invariant. $\square$