MATH 7290 Homework 1

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1 Problem 1

Work out the details in example 1.3.4 in Z. Yun's notes, assuming 1.3.3.

Proof. $G = SL_3$ and $e = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Standard basis v_1, v_2, v_3. Find $\mathcal{B}_e$.

By 1.3.3, $\mathcal{B}_e$ consists of flags $0 \subset V_1 \subset V_2 \subset V_3 = V$ such that $eV_i \subset V_{i-1}$ for $i = 1, 2, 3$. In particular, $eV_1 = 0$, so $V_1 \subset \ker e$. By inspection, $\ker e = \langle v_1, v_2 \rangle$. Thus, V_1 can be any line in $\langle e_1, e_2 \rangle$. There is a $\mathbb{P}^1$ worth of choices of generator u_1 for V_1, since we have a bijection of elements $[a : b] \in \mathbb{P}^1$ with non-zero vectors $av_1 + bv_2$ up to non-zero scalar multiple.

We now show that, away from the choice $V_1 = \langle v_1 \rangle$, the plane V_2 is fixed by the choice of V_1. Let $V_2 = \langle u_1, u_2 \rangle$. We have that $eV_2 \subset V_1$. Since $eu_1 = 0$, $eV_2 \subset V_1$ is equivalent to $eu_2 = cu_1$ for some scalar c. However, note that $eV = \langle v_1 \rangle$. Thus, if $V_1 \neq \langle v_1 \rangle$, in particular, if u_1 has a non-zero v_2 component, then we must have $c = 0$. Thus, $eu_2 = 0$, so $u_2 \in \ker e = \langle v_1, v_2 \rangle$. Since the basis elements of V_2 are both in $\langle v_1, v_2 \rangle$, we have $V_2 = \langle v_1, v_2 \rangle$.

Now consider the case $V_1 = \langle v_1 \rangle$. Without loss of generality, let $u_1 = v_1$. Write $u_2 = \alpha v_1 + \beta v_2 + \gamma v_3$. Again without loss of generality (this will not affect V_2), we can assume $\alpha = 0$. The condition $eu_2 = cu_1$ for some c is then satisfied for any choice of β and γ, as we will have $c = \gamma$. Once again, we have a $\mathbb{P}^1$ worth of choices. This shows $\mathcal{B}_e$ is a union of two copies of $\mathbb{P}^1$, as claimed in Yun's notes. $\square$

2 Problem 2

Work out the details in example 1.3.5 in Z. Yun's notes, assuming 1.3.3.

Proof. $G = SL_4$ and $e : v_3 \mapsto v_1 \mapsto 0, v_4 \mapsto v_2 \mapsto 0$. Standard basis $\{v_1, ..., v_4\}$. Find $\mathcal{B}_e$.

By 1.3.3, $\mathcal{B}_e$ consists of flags $0 \subset V_1 \subset ... \subset V_4 = V$ with $eV_i \subset V_{i-1}$. In particular, $eV_1 = 0$. Thus $V_1 \subset \ker e$. By construction, $\ker e = \langle v_1, v_2 \rangle$. As in Yun's notes, let H denote $\langle v_1, v_2 \rangle$.

If $V_2 = H$, then we have $eV_2 = 0 \subset V_1$. Again, by construction, we have $\mathrm{im} e = H$, so $eV_3 \subset \mathrm{im} e = H = V_2$. Thus, V_1 can be any line in H and V_3 can be any hyperplane in V containing H, and these choices give valid elements of $\mathcal{B}_e$. In this case, we have a bijection with $\mathbb{P}^1 \times \mathbb{P}^1$. As in Yun's notes, call this subset of $\mathcal{B}_e$ by C_1.

Now suppose $V_2 \neq H$. Let $V_1 = \langle av_1 + bv_2 \rangle$. Write $V_2 = \langle av_1 + bv_2, cv_1 + dv_2 + fv_3 + gv_4 \rangle$. Since we must have $eV_2 \subset V_1$, and we are assuming that f and g are not both 0 (from $V_2 \neq H$), we must have $[f : g] = [a : b]$. In particular, we can take $f = a, g = b$. The choice of $cv_1 + dv_2$ is defined up to scalar multiple of $av_1 + bv_2$. We will show that this description matches with Yun's claim after the next paragraph.

Continue with the assumption $V_2 \neq H$. We must have $eV_4 = \mathrm{im} e = H \subset V_3$, and by definition of flag we must have $V_2 \subset V_3$. Then V_3 is a three-dimensional space containing two distinct two-dimensional subspaces; there is only one possibility, namely that $V_3 = V_2 + H$. Note that this is also consistent with the condition $eV_3 \subset V_2$, since $e(V_2 + H) = eV_2 \subset V_1 \subset V_2$.

Now that we have seen that choosing V_1 and then V_2 in the case $V_2 \neq H$ determines V_3, we show that our data matches Yun's claim. The claim is that the elements of $\mathcal{B}_e$ satisfying $V_2 \neq H$ are in bijection with the total space of the line bundle $\mathcal{O}(2)$ over $\mathbb{P}^1$. This total space has the form $\{([x : y], \lambda x^2, \lambda xy, \lambda y^2) | [x : y] \in \mathbb{P}^1, \lambda \text{ scalar}\}$. To give a bijection between the two descriptions, we break it up into two cases.

In case one, assume that $V_1 = \langle v_2 \rangle$, i.e. that $[a : b] = [0 : 1]$. Then $cv_1 + dv_2$ can be reduced down to cv_1, for scalar c. On the other hand, restricting the line bundle $\mathcal{O}(2)$ to $[x : y] = [0 : 1]$ gives $\{([0 : 1], 0, 0, \lambda)\}$ for scalar λ. Thus we have a bijection in this case.

In case two, assume that $a \neq 0$, and without loss of generality, let $a = 1$. Then $cv_1 + dv_2$ can be taken to be $(d - bc)v_2$, where $d - bc$ can be made to be any scalar, call it h. On the other hand, restricting $\mathcal{O}(2)$ to $x \neq 0$ gives $\{([1 : y], \lambda, \lambda y, \lambda y^2)\}$. In particular, both sets in this case are in bijection with

$\mathbb{C}^2$, via the bijection $(b,h) \longleftrightarrow (y,\lambda)$.

To summarize, $\mathcal{B}_e$ is a union of two pieces. One is in bijection with $\mathbb{P}^1 \times \mathbb{P}^1$, representing the case where $V_2 = \langle v_1, v_2 \rangle$. When $V_2 \neq \langle v_1, v_2 \rangle$, there is a bijection with the total space of the line bundle $\mathcal{O}(2)$ on $\mathbb{P}^1$. $\square$

3 Problem 3

Show that the natural action of $SO(3)$ on S^2 is Hamiltonian as follows:

(i) Identify $\mathfrak{so}(3)$ with $\mathbb{R}^3$, where the Lie bracket on $\mathbb{R}^3$ is given by the cross product.

(ii) Now using the bilinear form $B(X, Y) = \operatorname{tr}(Y^T X)$ on $\mathfrak{so}(3)$, identify $\mathfrak{so}(3)$ with $\mathbf{so}(3)^*$.

(iii) Using the identifications in (i) and (ii), find the action on $SO(3)$ on $\mathbb{R}^3$ corresponding to the coadjoint action of $SO(3)$ on $\mathfrak{so}(3)^*$.

(iv) Write the moment map of this action.

Proof. (i) We identify $(x, y, z) \in \mathbb{R}^3$ with $\begin{pmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{pmatrix}$.

(ii) Given $X \in \mathfrak{so}(3)$, we have a functional $\hat{X} \in \mathfrak{so}(3)$ given by $\hat{X}(Y) = B(X, Y)$. This assignment is a linear map, and is injective, as we now argue. Under the identification with $\mathbb{R}^3$, computation shows that $B((x, y, z), (a, b, c)) = 2(ax + by + cz)$. In particular, $B(X, X) \neq 0$ if $X \neq 0$, so $\hat{X}$ is not the zero functional if $X \neq 0$. Since $\mathfrak{so}(3)$ is a finite dimensional vector space, its dual has the same finite dimension, and the injective linear map $X \mapsto \hat{X}$ is an isomorphism. That is to say, for a functional α, there is a unique X such that $\alpha = \hat{X}$. Therefore, we can identify $\mathfrak{so}(3)$ and $\mathfrak{so}(3)^*$. Let us make this identification more explicit.

Let $\{e_1, e_2, e_3\}$ be the basis of $\mathfrak{so}(3)$ corresponding to the standard basis of $\mathbb{R}^3$. Let $\{e^1, e^2, e^3\}$ be the dual basis of $\mathfrak{so}(3)^*$, i.e. the basis consisting of the unique functionals satisfying $e^i(e_j) = \delta_{i,j}$. Then we see that $\hat{e}_i = 2e^i$. This gives us an explicit form of the identification in terms of a basis of $\mathfrak{so}(3)^*$. That is, if $\alpha(e_i) = c_i$ for scalars c_1, c_2, c_3, then $\alpha = c_1 e^1 + c_2 e^2 + c_3 e^3$ is identified with $\frac{1}{2}(c_1 e_1 + c_2 e_2 + c_3 e_3) \in \mathfrak{so}(3)$, which is identified with $(\frac{1}{2}c_1, \frac{1}{2}c_2, \frac{1}{2}c_3) \in \mathbb{R}^3$.

(iii) The vector $(x_1, x_2, x_3) \in \mathbb{R}^3$ corresponds to the functional α satisfying $\alpha(e_i) = 2x_i$. For $g \in SO(3)$, the coadjoint action satisfies $(g \cdot \alpha)(Y) = \alpha(g^{-1} \cdot Y)$. For matrix Lie groups, the adjoint action is simply conjugation, i.e. $g^{-1} \cdot Y = g^{-1} Y g$. We must calculate $(g \cdot \alpha)(e_i) = \alpha(g^{-1} e_i g)$ for $i = 1, 2, 3$. Let $g = (g_{ij})$. By direct computation, we obtain

$$\begin{aligned}
\frac{1}{2}\alpha(g^{-1}e_1 g) &= (g_{22}g_{33} - g_{23}g_{32})x_1 + (g_{23}g_{31} - g_{21}g_{33})x_2 + (g_{21}g_{32} - g_{22}g_{31})x_3, \\
\frac{1}{2}\alpha(g^{-1}e_2 g) &= (g_{13}g_{32} - g_{12}g_{33})x_1 + (g_{11}g_{33} - g_{13}g_{31})x_2 + (g_{12}g_{31} - g_{11}g_{32})x_3, \\
\frac{1}{2}\alpha(g^{-1}e_3 g) &= (g_{12}g_{23} - g_{13}g_{22})x_1 + (g_{13}g_{21} - g_{11}g_{23})x_2 + (g_{11}g_{22} - g_{12}g_{21})x_3.
\end{aligned}$$

The coordinates of $g \cdot (x_1, x_2, x_3)$ in $\mathbb{R}^3$ are given by the right hand side of the above equations. In fact, these coordinates can be made simpler by using the $SO(3)$ structure. We know that $\det g = 1$ and $g^T = g^{-1}$, so we have

$$\begin{pmatrix} g_{11} & g_{21} & g_{31} \\ g_{12} & g_{22} & g_{32} \\ g_{13} & g_{23} & g_{33} \end{pmatrix} = \begin{pmatrix} g_{22}g_{33} - g_{23}g_{32} & g_{13}g_{32} - g_{12}g_{33} & g_{12}g_{23} - g_{13}g_{22} \\ g_{23}g_{31} - g_{21}g_{33} & g_{11}g_{33} - g_{13}g_{31} & g_{13}g_{21} - g_{11}g_{23} \\ g_{21}g_{32} - g_{22}g_{31} & g_{12}g_{31} - g_{11}g_{32} & g_{11}g_{22} - g_{12}g_{21} \end{pmatrix}.$$

As a result, the coordinates of $g \cdot (x_1, x_2, x_3)$ are

$$\begin{aligned} &g_{11}x_1 + g_{12}x_2 + g_{13}x_3, \\ &g_{21}x_1 + g_{22}x_2 + g_{23}x_3, \\ &g_{31}x_1 + g_{32}x_2 + g_{33}x_3. \end{aligned}$$

Thus, the coadjoint action of $SO(3)$ is identified with the ordinary action of $SO(3)$ on $\mathbb{R}^3$ by matrix multiplication.

(iv) To find the moment map, we first determine the infinitesimal action of $\mathfrak{so}(3)$ on S^2 (or simply $\mathbb{R}^3$). For instance, how does the basis element e_1 act on $(x, y, z) \in \mathbb{R}^3$? By definition, this is $\frac{d}{dt}|_{t=0}[\exp(te_1)(x, y, z)]$. Using the standard matrix exponential, we find

$$\exp(te_1) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix},$$

and thus $e_1 \cdot (x, y, z) = (0, -z, y)$. In other words, the fundamental vector field $e_1^\#$ is $-z\partial_y + y\partial_z$. Similar computations lead to $e_2^\# = z\partial_x - x\partial_z$ and $e_3^\# = -y\partial_x + x\partial_y$.

Now, if the action truly is Hamiltonian, we can find functions f_1, f_2, f_3 such that $df_i = \omega(e_i^\#, -)$. First, recall the symplectic form on S^2 is

$$\begin{aligned} \omega &= xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \\ &= x(dy \otimes dz - dz \otimes dy) + y(dz \otimes dx - dx \otimes dz) + z(dx \otimes dy - dy \otimes dx). \end{aligned}$$

Computing $\omega(e_1^\#, -)$ explicitly gives $(y^2 + z^2)dx - xydy - xzdz$. This is not integrable on $\mathbb{R}^3$, but notice that we only care about the sphere, where $x^2 + y^2 + z^2 = 1$. Thus $xdx + ydy + zdz = 0$, giving $x^2dx = -xydy - xzdz$. Thus $\omega(e_1^\#, -) = (x^2 + y^2 + z^2)dx = dx$. Then we can choose $f_1 = x$. Similar computation gives $f_2 = y, f_3 = z$. The f_i define a Hamiltonian function $H : \mathfrak{so}(3) \to \mathcal{O}(S^2)$. The moment map is then given by $\mu(m)(X) = H(X)(m)$. On the basis vectors, this gives $\mu((x, y, z))(e_1) = x$, $\mu((x, y, z))(e_2) = y$, $\mu((x, y, z))(e_1) = z$. Thus, μ sends a point (x, y, z) to the functional which we have identified with $(\frac{1}{2}x, \frac{1}{2}y, \frac{1}{2}z) \in \mathbb{R}^3$. (Note: I know morally this should be the identity/inclusion map. I don't know if I made a mistake somewhere, or if the bilinear form should have had the $\frac{1}{2}$ built into it.) $\square$