MATH 7250 Homework 5

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1 Problem 1

Let λ be a partition of n viewed as a Young diagram.

(a) Show that $\text{Res}_{S_{n-1}}^{S_n} V_\lambda$ is the direct sum of V_μ's, where μ runs over all Young diagrams obtained by removing a box from λ.

Proof. Note: this proof is essentially from Bruce Sagan's book on symmetric groups, with more details as necessary.

First we show that the boxes of the Young diagram λ which can be removed to obtain a new Young diagram are exactly those where the label n can be placed in a standard Young tableaux of shape λ. Indeed, a standard tableaux has increasing rows and columns, so the n must be in a box with nothing to the right or below. Any box with something to the right or below cannot be removed to make a Young diagram, because there will be a "hole" in the diagram. However, the boxes with nothing to the right are below can be removed with no problem.

This result implies that the number of standard λ-tableaux equals the sum of the number of standard μ-tableaux, with the sum ranging over μ obtained from removing a box from λ. Recall that the dimension of V_λ is the number of standard λ-tableaux, so this result implies that $\dim V_\lambda = \sum_\mu V_\mu$.

Let $r_1, ..., r_k$ be the rows of λ which have a box that can be removed. For each $i = 1, ..., k$, let λ^i be the partition obtained from removing the box at the end of row r_i. Furthermore, if λ-tableau has n in row r_i, let t_i be the corresponding λ^i-tableau obtained by removing the box containing n. Now, for a subspace $W \subset V$, we have $V = W \oplus (V/W)$. Thus, to show that $\text{Res}V_\lambda = \oplus_i V_{\lambda^i}$, it suffices to find a chain of subspaces

$$0 = V^0 \subset ... \subset V^k = V_\lambda$$

such that $V^i/V^{i-1} \cong V_{\lambda^i}$ as S_{n-1} modules. To that end, we define V^i as the span of the polytabloids e_t, where t is a standard λ-tableau with the n in some row $r_1, ..., r_i$. We note that $V^k = V_\lambda$ because any standard tableau must have n

in one of the rows $r_1, ..., r_k$, so we have all the polytabloids needed to span V_λ.

Let U_λ be the permutation module, i.e. the span of λ-tabloids. Let $\theta_i : U_\lambda \to U_{\lambda^i}$ be a linear map defined by $\theta_i(\{t\}) = \{t_i\}$ if n is in row r_i of t, and $\theta_i(\{t\}) = 0$ otherwise. Since the map is defined in terms of where the n is in t, this is clearly a map of S_{n-1}-modules, where we are saying S_{n-1} sits in S_n as the stabilizer of $n \in \{1, ..., n\}$.

If t is a standard λ-tableau with n in row r_j, then any tabloid appearing in the expansion of e_t, i.e. one obtained by applying permuting the columns, will have the n in a row r_k with $k \leq j$. This is because the n in t has nothing below it, so permuting the column cannot decrease the row position of the n. Then, for such t, we have $\theta_i(e_t) = e_{t_i}$ if $i = j$ and $\theta_i(e_t) = 0$ if $j < i$. As a result, $\theta_i(V^i) = S_{\lambda^i}$ and $V^{i-1} \subset \ker \theta_i$. Then we have a chain

$$0 = V^0 \subset V^1 \cap \ker \theta_1 \subset V_1 \subset ... \subset V^k = V_\lambda.$$

By isomorphism theorem, $\dim \left(V^i / V^i \cap \ker \theta_i\right) = \dim \theta_i V^i = \dim V_{\lambda^i}$. We noticed earlier that $\dim V_\lambda = \sum_i V_{\lambda^i}$, which means that the quotients $V^i \cap \ker \theta_i / V^{i-1}$ are 0-dimensional, i.e. that $V^{i-1} = V^i \cap \ker \theta_i$. Thus $V^i / V^{i-1} \cong V_{\lambda^i}$ as desired. □

(b) Show that $\mathrm{Ind}_{S_n}^{S_{n+1}} V_\lambda$ is the direct sum of V_ν's, where ν runs over all Young diagrams obtained by adding a box to λ.

Proof. Note that ν is obtained by adding a box to λ if and only if λ is obtained by removing a box from ν. From part (a) and Frobenius reciprocity, i.e. $\dim \mathrm{Hom}(\mathrm{Ind} V_\lambda, V_\nu) = \dim \mathrm{Hom}(V_\lambda, \mathrm{Res} V_\nu)$, we get the desired result. □

2 Problem 3

Let λ be a partition of n, and let V_λ be the corresponding irrep of S_n.

(a) Given complex numbers $a_1, ..., a_N$, show that

$$\det\left(\prod_{k=1}^{N-i}(a_j - k)\right) = \prod_{1\le i<j\le N}(a_i - a_j).$$

Proof. Note that this determinant, call it D, is divisible by $\Delta = \prod_{1\le i<j\le N}(a_i - a_j)$, since $\Delta = 0$ exactly when some $a_i = a_j$, in which case the matrix has two identical columns and thus 0 determinant. The degree of Δ is $N(N-1)/2$. The highest degree which appears in the expansion of D is $N(N-1)/2$, as one can see by multiplying the diagonal elements. Thus $D = c\Delta$ for some constant c. To determine c, let $a_i = N+1-i$ for all i. The matrix becomes lower triangular, so that

$$D = \prod_{i=1}^{N}\prod_{k=1}^{N-i}(N+1-i-k) = \prod_{i=1}^{N}(N-i)! = \prod_{i=1}^{N-1}(N-i)!.$$

On the other hand, for these values of a_i we have

$$\Delta = \prod_{1\le i<j\le N}(j-i) = \prod_{i=1}^{N-1}\prod_{j=i+1}^{N}(j-i) = \prod_{i=1}^{N-1}\prod_{j-i=1}^{N-i}(j-i) = \prod_{i=1}^{N-1}(N-i)!.$$

Since $D = \Delta \neq 0$ for this choice of a_i, and $D = c\Delta$ for some fixed constant in general, it must be that $D = \Delta$ for all a_i. □

(b) Let N be an integer at least as large as the number of parts of λ, and set $\ell_j = \lambda_j + N - j$. Apply the Frobenius character formula to show that

$$\dim V_\lambda = \frac{n!}{\prod_j \ell_j!}\prod_{1\le i<j\le N}(\ell_i - \ell_j).$$

Proof. The following proof is adapted from Fulton and Harris. Recall that the dimension of a representation is exactly its character evaluated at the trivial conjugacy class. The trivial conjugacy class in S_n corresponds to the partition (1,...,1), which has $i = (n, 0, ..., 0)$. Thus we want the coefficient of x^ℓ (abbreviated monomial) in $(x_1 + ... + x_N)^n \prod_{1\le i<j\le N}(x_i - x_j)$. But $\prod_{1\le i<j\le N}(x_i - x_j)$ is the Vandermonde determinant, so it is equal to $\sum_{\sigma\in S_N}(-1)^\sigma x_N^{\sigma(1)-1}\cdots x_1^{\sigma(N)-1}$. On the other hand, we have the multinomial expansion $(x_1 + ... + x_N)^n = \sum \frac{n!}{\prod r_j!}x^r$, with the sum over $(r_1, ..., r_N) \in \mathbb{Z}^N_{\le 0}$ with $\sum r_j = n$. To get x^l, we must then pair a term from the σ sum with a term from the multinomial sum such that $r_j + \sigma(N-j+1) - 1 = \ell_j$. Note that this requires $\sigma(N-j+1) - 1 \le \ell_j$,

which is a restraint we put on σ for the following sum. Then we get

$$\dim V_\lambda = \sum_\sigma (-1)^\sigma \frac{n!}{\prod_j (\ell_j - \sigma(N-j+1)+1)!}$$

$$= \frac{n!}{\prod_j \ell_j!} \sum_\sigma (-1)^\sigma \prod_j \ell_j(\ell_j - 1)\cdots(\ell_j - \sigma(N-j+1)+2).$$

This sum is the determinant of the matrix in part (a) with $a_i = \ell_i + 1$, so we have

$$\dim V_\lambda = \frac{n!}{\prod_j \ell_j!} \prod_{1 \le i < j \le N} (\ell_i - \ell_j)$$

as desired. ☐

(c) Show that

$$\dim V_\lambda = \frac{n!}{\prod \text{hook lengths}},$$

where the product is over all boxes in the Young diagram for λ.

Proof. From part (b), it suffices to show that

$$\prod \text{hook lengths} = \frac{\prod_j \ell_j!}{\prod_{1\le i<j\le N}(\ell_i - \ell_j)}.$$

But

$$\frac{\prod_j \ell_j!}{\prod_{1\le i<j\le N}(\ell_i - \ell_j)} = \prod_{1\le i\le N} \frac{\ell_i!}{\prod_{i<j\le N}(\ell_i - \ell_j)},$$

so it suffices to show that the product of the hook lengths in the ith row is $\ell_i!/\prod_{i<j\le N}(\ell_i - \ell_j)$. In fact, notice that the hook length of a box doesn't depend on any of the rows above it. In particular, we can argue that if we know the product of the hook lengths in the top row, then we know the product of the hook lengths in the next row, and we can induct to get the rest of the rows. Indeed, let λ' be the partition of $n - \lambda_1$ obtained by removing the top row of λ, so that $\lambda'_j = \lambda_{j+1}$. Since the number of parts of λ' is one less than the number of parts of λ, we take $N' = N-1$. Then $\ell'_j = \lambda'_j + N' - j = \lambda_{j+1} + N - (j+1) = \ell_{j+1}$. Thus the product of the hook lengths on the second row of λ, i.e. the top row of λ', is

$$\frac{\ell'_1!}{\prod_{1<j\le N'}(\ell'_1 - \ell'_j)} = \frac{\ell_2!}{\prod_{1<j\le N-1}(\ell_2 - \ell_{j+1})} = \frac{\ell_2!}{\prod_{2<j\le N}(\ell_2 - \ell_j)}.$$

We can repeat this process since there are finitely many rows in a partition, so all we need to do is finally prove the product of the hook lengths along the top row.

First, we make another reduction. We have that N is greater than or equal to k, the number of parts of λ. Suppose we have the hook length formula for $N = k$, and we will show it works for $N > k$. Let $m = N - k$, and let $\ell'_j = \lambda_j + k - j$ for $j \in \{1, ..., k\}$. Then $\ell_j = \ell'_j + m$ for $j \in \{1, ..., k\}$, and $\ell_j = N - j$ for $j \in \{k+1, ..., N\}$. Notice that $\ell_1 - \ell_j = \ell'_1 - \ell'_j$ for $j \in \{1, ..., k\}$, since the m cancels. For $j \in \{k+1, ..., N\}$ we have $\ell_1 - \ell_j = \ell'_1 + m - (N - j) = \ell'_1 + m - (k + m - j) = \ell'_1 + j - k$. Then

$$\frac{\ell_1!}{\ell'_1!} = \frac{(\ell'_1 + m)!}{\ell'_1!} = \prod_{s=1}^{m} (\ell'_1 + s) = \prod_{k < j \leq N} (\ell_1 - \ell_j).$$

Thus

$$\begin{aligned} \frac{\ell_1!}{\prod_{1<j\leq N}(\ell_1 - \ell_j)} &= \frac{\ell_1!}{\ell'_1!} \frac{\ell'_1!}{\prod_{1<j\leq k}(\ell_1 - \ell_j)} \frac{1}{\prod_{k<j\leq N}(\ell_1 - \ell_j)} \\ &= \frac{\ell'_1!}{\prod_{1<j\leq k}(\ell'_1 - \ell'_j)}. \end{aligned}$$

Now, we must show that for $N = k$ we have the desired product. But I don't know how to do this. □

3 Problem 5

Let F be a field.

(a) Show that the derived group $[GL_n(F), GL_n(F)] \subset SL_n(F)$.

Proof. It suffices to show that the group commutator of two invertible matrices has determinant one: if all the commutators are contained in $SL_n(F)$, then they generate a group contained in $SL_n(F)$. This is true because the determinant is a group homomorphism $GL_n(F) \to F^\times$, so $\det(ABA^{-1}B^{-1}) = \det(A)\det(B)\det(A)^{-1}\det(B)^{-1} = 1$. $\square$

(b) If $n \geq 3$, show that any elementary matrix is the commutator of elementary matrices.

Proof. Let e_{ij} denote the matrix with a 1 in the (i,j) position and 0 elsewhere. Note that $e_{ij}e_{kl} = \delta_{jk}e_{il}$, where δ_{jk} is the Kronecker delta symbol. An elementary matrix can be expressed as $I + xe_{ij}$ where I is identity and $i \neq j$. The inverse of such a matrix is $I - xe_{ij}$ since $(I+xe_{ij})(I-xe_{ij}) = I + xe_{ij} - xe_{ij} = I$. Since $n \geq 3$, there is some k which is not equal to i and not equal to j. Then

$$\begin{aligned}
[I + xe_{ik}, I + e_{kj}] &= (I + xe_{ik})(I + e_{kj})(I - xe_{ik})(I - e_{kj}) \\
&= (I + xe_{ik} + e_{kj} + xe_{ij})(I - xe_{ik} - e_{kj} + xe_{ij}) \\
&= I - xe_{ik} - e_{kj} + xe_{ij} + xe_{ik} - xe_{ij} + e_{kj} + xe_{ij} = I + xe_{ij}.
\end{aligned}$$

$\square$

(c) If $n = 2$ and $|F| \geq 3$, show that any elementary matrix is the commutator of a diagonal matrix and an elementary matrix.

Proof. We have

$$\begin{aligned}
\left[\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right] &= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}^{-1} \\
= \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -x \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 2 & 2x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{-x}{2} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},
\end{aligned}$$

and similarly

$$\left[\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right] = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}.$$

$\square$

(d) Show that $[GL_n(F), GL_n(F)] = SL_n(F)$ unless $n = 2$ and $|F| = 2$. What happens for $GL_2(\mathbb{F}_2)$?

Proof. Parts (b) and (c) show that, unless $n = 2$ and $|F| = 2$, all elementary matrices are in $[GL_n(F), GL_n(F)]$. But elementary matrices generate $SL_n(F)$, so $[GL_n(F), GL_n(F)] = SL_n(F)$.

Note that $GL_2(\mathbb{F}_2) = SL_2(\mathbb{F}_2)$, since the only nonzero element of $\mathbb{F}_2^\times$ is 1. Since there are only sixteen 2x2 matrices with entries in $\mathbb{F}_2$, we can write them all down and find that there are six with nonzero determinant. Let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tau = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. We find that $\sigma^2 = I = \tau^3$ and $\sigma\tau\sigma = \tau^2$, which shows that $GL_2(\mathbb{F}_2) \cong S_3$. Now, notice that in the commutator $[A, B] = ABA^{-1}B^{-1}$, we have a conjugate of B and an inverse of B. If $B = I$, then $[A, B] = I$. If B is a transposition, then so are ABA^{-1} and B^{-1}. The product of two transpositions in S_3 is either I or a 3-cycle. If B is a 3-cycle, then so are ABA^{-1} and B^{-1}. The product of two 3-cycles in S_3 is either I or a 3-cycle. Thus, in any case, $[A, B]$ is either I or a 3-cycle, meaning that $[GL_2(\mathbb{F}_2), GL_2(\mathbb{F}_2)] \cong \mathbb{Z}/3\mathbb{Z}$ is a proper subgroup of $SL_2(\mathbb{F}_2)$. $\square$