MATH 7250 Homework 4

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1 Problem 2

Let θ be the linear automorphism of $\mathbb{C}S_n$ defined by $\pi \mapsto (-1)^\pi \pi$ for $\pi \in S_n$.

(a) Show that θ is an algebra automorphism and that if (V, ρ) is a representation of S_n, then $(V, \rho \circ \theta) \cong (V, \rho) \otimes \text{sgn}$.

Proof. We are given that θ is a linear automorphism. That it is an algebra automorphism is simply due to the fact that the sign of permutations is multiplicative, so that $\theta(\pi\sigma) = (-1)^{\pi\sigma}\pi\sigma = (-1)^\pi \pi (-1)^\sigma \sigma = \theta(\pi)\theta(\sigma)$.

Let $\eta : (V, \rho \circ \theta) \to (V, \rho) \otimes \text{sgn}$ be the map $v \mapsto v \otimes 1$. This is clearly a linear isomorphism. It is also a map of representations, because $(\rho \circ \theta)(\pi)v = (-1)^\pi \rho(\pi)v$ and $\pi(v \otimes 1) = \rho(\pi)v \otimes (-1)^\pi = (-1)^\pi \rho(\pi)v \otimes 1$. Thus η is an isomorphism of representations. $\square$

(b) If λ is a partition, let λ^t denote the conjugate partition. Show that $V_{\lambda^t} \cong V_\lambda \otimes \text{sgn}$.

Proof. Take a λ-tableau T and consider its conjugate T^t. Then the row subgroup $P_T \leq S_n$ for T is the column subgroup Q_{T^t} for T^t, and similarly $Q_T = P_{T^t}$. Then

$$\theta(c_T) = \theta\left(\sum_{g \in P_T, h \in Q_T} \text{sgn}(h) gh\right)$$

$$= \sum_{g \in P_T, h \in Q_T} \text{sgn}(h)\text{sgn}(g)\text{sgn}(h)\theta(g)\theta(h) = \sum_{g \in P_T, h \in Q_T} \text{sgn}(g)\theta(g)\theta(h)$$

$$= \sum_{g \in Q_{T^t}, h \in P_{T^t}} \text{sgn}(g)\theta(g)\theta(h) = b_{T^t} a_{T^t}.$$

Thus, from part (a), we have $\mathbb{C}S_n b_{T^t} a_{T^t} \cong \mathbb{C}S_n a_T b_T = V_\lambda$. However, from HW Problem 1 (which I will take for granted), we know that $\mathbb{C}S_n b_{T^t} a_{T^t} \cong \mathbb{C}S_n a_{T^t} b_{T^t} = V_{\lambda^t}$, so we are done. $\square$

2 Problem 3

(a) If λ is any partition of n, show that $\operatorname{Ind}_{A_n}^{S_n}\operatorname{Res}_{A_n}^{S_n}V_\lambda \cong V_\lambda \oplus V_{\lambda^t}$.

Proof. It is a general fact that $\operatorname{Ind}_H^G\operatorname{Res}_H^G V \cong V \otimes P$, where P is the permutation representation of the cosets G/H. This follows from Homework 3 Problem 3(c), with $W = 1_H$. In our case, $G = S_n, H = A_n$, there are only two cosets in G/H, so that P is a two-dimensional representation. One coset corresponds to the even permutations, and the other corresponds to the odd permutations. Choose for P the basis e_0, e_1 representing the even and odd cosets respectively. In this basis, any even permutation acts by the identity, while any odd permutation acts by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Thus, the character of P has 2's for even permutations and 0's for odd permutations. This is the same as the sum of characters of the trivial and sign representations, so we must have $P \cong 1 \oplus \operatorname{sgn}$. Thus,

$$\operatorname{Ind}_{A_n}^{S_n}\operatorname{Res}_{A_n}^{S_n}V_\lambda \cong V_\lambda \otimes (1 \oplus \operatorname{sgn}) \cong V_\lambda \oplus (V_\lambda \otimes \operatorname{sgn}) = V_\lambda \oplus V_{\lambda^t},$$

where we have used the isomorphism $V_{\lambda^t} \cong V_\lambda \otimes \operatorname{sgn}$ from the previous problem. $\square$

(b) Show that

$$\dim \operatorname{Hom}_{A_n}(\operatorname{Res}_{A_n}^{S_n}V_\lambda, \operatorname{Res}_{A_n}^{S_n}V_\mu) = \begin{cases} 2 & \text{if } \lambda = \mu = \mu^t \\ 1 & \text{if } \lambda = \mu \text{ or } \mu^t \text{ and } \mu \neq \mu^t \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since Res and Ind are adjoint, we have

$$\begin{aligned} \dim \operatorname{Hom}_{A_n}(\operatorname{Res}_{A_n}^{S_n}V_\lambda, \operatorname{Res}_{A_n}^{S_n}V_\mu) &= \dim \operatorname{Hom}_{S_n}(\operatorname{Ind}_{A_n}^{S_n}\operatorname{Res}_{A_n}^{S_n}V_\lambda, V_\mu) \\ &= \dim \operatorname{Hom}_{S_n}(V_\lambda \oplus V_{\lambda^t}, V_\mu). \end{aligned}$$

By Schur's lemma, this dimension counts the number of times V_μ appears in $V_\lambda \oplus V_{\lambda^t}$, which gives the desired cases. $\square$

(c) Show that if $\lambda \neq \lambda^t$, then $\operatorname{Res}_{A_n}^{S_n}V_\lambda$ is irreducible, while if $\lambda = \lambda^t$, we have $\operatorname{Res}_{A_n}^{S_n}V_\lambda = W_\lambda \oplus W'_\lambda$, where W_λ and W'_λ are conjugate, nonisomorphic irreps.

Proof. Recall that if $V = \bigoplus k_i V_i$ is a decomposition of a representation into irreducibles, then $\dim \operatorname{Hom}_G(V, V) = \sum k_i^2$. This follows from Schur's lemma. From above, if $\lambda \neq \lambda^t$, we have $\dim \operatorname{Hom}_{A_n}(\operatorname{Res}_{A_n}^{S_n}V_\lambda, \operatorname{Res}_{A_n}^{S_n}V_\lambda) = 1$, which is enough to conclude irreducibility. On the other hand, if $\lambda = \lambda^t$, then $\dim \operatorname{Hom}_{A_n}(\operatorname{Res}_{A_n}^{S_n}V_\lambda, \operatorname{Res}_{A_n}^{S_n}V_\lambda) = 2$. This implies that $\operatorname{Res}_{A_n}^{S_n}V_\lambda$ is a sum of non-isomorphic irreps, say W_λ and W'_λ, since the only way to express 2 as a sum of squares is $1^2 + 1^2$. Since $\operatorname{Res}_{A_n}^{S_n}V_\lambda$ is self-conjugate, either W_λ and W'_λ are conjugate to each other, or they are both self-conjugate. If they are both self-conjugate, then this lifts to a decomposition of V_λ, which is irreducible. Thus they are conjugate to each other, as desired. $\square$

(d) Show that every irrep of A_n is isomorphic to exactly one of the irreps defined in the previous part.

Proof. Consider two distinct partitions λ, μ of n. Then

$$\dim \mathrm{Hom}_{A_n}(\mathrm{Res}^{S_n}_{A_n} V_\lambda, \mathrm{Res}^{S_n}_{A_n} V_\mu) = 0.$$

If $\lambda \neq \lambda^t$ and $\mu \neq \mu^t$, then $\mathrm{Res}V_\lambda$ and $\mathrm{Res}V_\mu$ are both irreducible, and the dimension formula shows they are not isomorphic. Similarly, if $\mu = \mu^t$ and $\lambda \neq \lambda^t$, then neither W_μ nor W'_μ are isomorphic to $\mathrm{Res}V_\lambda$. If λ and μ are both self-conjugate, then $W_\lambda, W'_\lambda, W_\mu, W'_\mu$ are all pairwise non-isomorphic, since an isomorphism between one involving λ and one involving μ would contribute to the dimension formula above. Thus, all the irreps we have obtained are non-isomorphic. We obtained 2 for each self-conjugate partition of n, and 1 for each unordered pair $\{\lambda, \lambda^t\}$ where $\lambda \neq \lambda^t$.

To show that these are all of the irreps, we use reciprocity. Let W be an irrep of A_n. If it is disjoint from all the irreps above, then we have $\dim \mathrm{Hom}(W, \mathrm{Res}V_\lambda) = 0$ for all partitions λ. By Frobenius reciprocity, this means $\dim \mathrm{Hom}(\mathrm{Ind}W, V_\lambda) = 0$ for all λ, which is impossible, since the V_λ are all of the irreps of S_n and there must be at least one sitting in $\mathrm{Ind}W$. $\square$

3 Problem 6

Let F be an arbitrary field. Let $\mathcal{T}_\lambda$ be the set of Young tabloids of shape λ, and let $U_\lambda = F[\mathcal{T}_\lambda]$ be the corresponding permutation module. Define the Specht module S^λ as the submodule of U_λ spanned by the polytabloids e_T as T ranges over all λ-tableaux.

(a) Show that if S and T are two λ-tableaux, then $b_T\{S\} = 0$ if two numbers in the same row of S are in the same column of T, and $b_T\{S\} = \pm e_T$ otherwise. Conclude that $b_T U_\lambda \subset F e_T$.

Proof. Recall that $b_T = \sum_{g \in Q_T} (-1)^g g$, where Q_T is the subgroup of S_n which preserves columns of T. First suppose that $\lambda = (n)$. Then S and T are obviously equivalent, so $b_T\{S\} = b_T\{T\} = e_T$. If $\lambda \neq (n)$, then T has a column with at least two elements. Let a, b be two distinct elements in the same column of T. Then (ab) is an odd element of Q_T, and in particular induces an involution of Q_T which swaps even and odd elements by right multiplication. We can then write

$$b_T = \sum_{\substack{g \in Q_T \\ g \text{ even}}} g - \sum_{\substack{g \in Q_T \\ g \text{ odd}}} g = \sum_{\substack{g \in Q_T \\ g \text{ even}}} g - \sum_{\substack{g \in Q_T \\ g \text{ even}}} g(ab)$$

$$= \left(\sum_{\substack{g \in Q_T \\ g \text{ even}}} g \right) (1 - (ab)).$$

Thus we can write $b_T = y(1 - (ab))$ for $y \in FS_n$. If a, b are in the same row of S, then $(ab)S$ and S are equivalent, so that $b_T\{S\} = y(1 - (ab))\{S\} = y\{S\} - y\{(ab)S\} = 0$.

Now suppose that any two numbers in the same column of T are not in the same row of S. Then there is some $\pi \in Q_T$ such that $\{S\} = \pi\{T\}$, so that $b_T\{S\} = b_T \pi\{T\} = (-1)^\pi b_T\{T\} = \pm e_T$.

Finally, if $\{S_1\}, ..., \{S_k\}$ is a basis for U_λ, ordered so that for some r, $b_T\{S_i\} = \pm e_T$ for $i \leq r$ and $b_T\{S_i\} = 0$ for $i > r$. Then

$$b_T(a_1\{S_1\} + ... + a_k\{S_k\}) = (\pm a_1 \pm ... \pm a_r)e_T,$$

so $b_T U_\lambda \subset F e_T$ as desired. $\square$

(b) Define a symmetric bilinear form on U_λ via $\langle \{S\}, \{T\} \rangle = \delta_{\{S\},\{T\}}$. Show that this form is S_n-invariant and satisfies $\langle b_T u, v \rangle = \langle u, b_T v \rangle$ for any Young tableau T.

Proof. By linearity, it suffices to show S_n-invariance when the form is taken on basis elements $\{S\}$ and $\{T\}$. Of course, this just follows from the well-definedness of the S_n action on tabloids. If $\{S\} = \{T\}$, i.e. S and T are row-equivalent, then πS and πT are row equivalent for any $\pi \in S_n$. If S and T are not row equivalent, then πS and πT cannot be row equivalent, since, if that were the case, we could apply π^{-1} to conclude that S and T are row equivalent.

Again, by linearity, it suffices to show that $\langle b_T\{S_1\}, \{S_2\}\rangle = \langle \{S_1\}, b_T\{S_1\}\rangle$. We have

$$\begin{aligned}\langle b_T\{S_1\}, \{S_2\}\rangle = \langle \sum_{g \in Q_T} (-1)^g g\{S_1\}, \{S_2\}\rangle &= \sum_{g \in Q_T} (-1)^g \langle g\{S_1\}, \{S_2\}\rangle \\ &= \sum_{g \in Q_T} (-1)^g \langle \{S_1\}, g^{-1}\{S_2\}\rangle,\end{aligned}$$

where at the last step we have used the S_n invariance of the form. Since the sign of g is the same as the sign of g^{-1}, we can relabel the sum and use linearity to get the desired result. □

(c) Show that if $M \subset U_\lambda$ is a submodule, then either $S^\lambda \subset M$ or $(S^\lambda)^\perp \supset M$.

Proof. If $a \in M$ and T is a λ-tableaux, then $b_T a$ is in Fe_T by part (a), and in M by definition of submodule. If it is possible to choose a, T such that $b_T a \neq 0$, then $e_T \in M$, and therefore $\pi e_T = e_{\pi T} \in M$ for all $\pi \in S_n$. Thus, $S^\lambda \subset M$. On the other hand, if for all a, T we have $b_T a = 0$, then $0 = \langle b_T a, \{T\}\rangle = \langle a, b_T\{T\}\rangle = \langle a, e_T\rangle$, so $a \in (S^\lambda)^\perp$. Thus $M \subset (S^\lambda)^\perp$. □

(d) Show that $S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$ is either 0 or irreducible. In the latter case, show that $S^\lambda \cap (S^\lambda)^\perp$ is the unique maximal submodule of S^λ.

Proof. From part (c), any submodule $M \subset S^\lambda$ either is S^λ or is contained in $S^\lambda \cap (S^\lambda)^\perp$. That statement exactly implies that $S^\lambda \cap (S^\lambda)^\perp$ is the unique maximal ideal, if it is not all of S^λ. If it is all of S^λ, then the quotient is clearly 0. Finally, since submodules L of a quotient M/N correspond to submodules of M containing N, we see that $S^\lambda/(S^\lambda \cap (S^\lambda)^\perp)$ cannot have any nontrivial submodules, since the only submodules of S^λ containing $S^\lambda \cap (S^\lambda)^\perp$ are $S^\lambda \cap (S^\lambda)^\perp$ and S^λ. □