MATH 7250 Homework 3

Andrea Bourque

September 2022

1 Problem 3

Let H be subgroup of G and let γ and δ be class functions on H and G respectively.

(a) Define a function on G via

$$\operatorname{Ind}_H^G(\gamma)(g) = \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}gt \in H}} \gamma(t^{-1}gt).$$

Show directly that $\operatorname{Ind}_H^G(\gamma)$ is a class function on G.

Proof. Let $x, g \in G$. Then

$$\begin{aligned} \operatorname{Ind}_H^G(\gamma)(x^{-1}gx) &= \frac{1}{|H|} \sum_{\substack{t \in G \\ t^{-1}x^{-1}gxt \in H}} \gamma(t^{-1}x^{-1}gxt) \\ &= \frac{1}{|H|} \sum_{\substack{t' \in G \\ t'^{-1}gt' \in H}} \gamma(t'^{-1}gt') = \operatorname{Ind}_H^G(\gamma)(g), \end{aligned}$$

where $t' = xt$. Reindexing the sum is valid because $x : G \to G$ is a bijection. □

(b) Let C be a conjugacy class of G and let $C_1, ..., C_s$ be the (possibly empty) collection of H-conjugacy classes into which $C \cap H$ decomposes. Show that

$$\operatorname{Ind}_H^G(\gamma)(C) = \frac{[G:H]}{|C|} \sum_{i=1}^{s} |C_i| \gamma(C_i).$$

Proof. Fix a representative $g \in C$. The sum defining $\operatorname{Ind}_H^G(\gamma)$ picks out elements $t^{-1}gt = x \in C \cap H$. There may overcounting corresponding to the situation where $t_1^{-1}gt_1 = t_2^{-1}gt_2$ for distinct $t_1, t_2 \in G$. This means $t_1t_2^{-1}$ is in the stabilizer of g, which by orbit-stabilizer has order $|G|/|C|$. Then all the elements of $C \cap H$ are overcounted by $|G|/|C|$. Thus,

$$\mathrm{Ind}_H^G(\gamma)(C) = \frac{1}{|H|}\frac{|G|}{|C|}\sum_{x \in C\cap H}\gamma(x) = \frac{[G:H]}{|C|}\sum_{x\in C\cap H}\gamma(x).$$

Since γ is a class function, the sum can be broken up into a sum over the H-conjugacy classes in $C \cap H$. Grouping the terms $\gamma(x)$ for $x \in C_i$ gives $|C_i|\gamma(C_i)$. This gives the desired result. □

(c) Let V and W be representations of G and H respectively. Show that $\mathrm{Ind}_H^G(W) \otimes V \cong \mathrm{Ind}_H^G(W \otimes \mathrm{Res}_H^G(V))$.

Proof. Recall that $\mathrm{Ind}_H^G = \mathbb{C}G \otimes_{\mathbb{C}H}$. Then define a map $\eta : \mathrm{Ind}_H^G(W \otimes \mathrm{Res}_H^G(V)) \to \mathrm{Ind}_H^G(W) \otimes V$ by $\eta(a \otimes (w \otimes v)) = (a \otimes w) \otimes av$ where $a \in \mathbb{C}G, w \in W, v \in V$, and extend by linearity. We should check that this is well-defined with the $\mathbb{C}H$ tensor product. Namely, for any $h \in H$, we have $a \otimes (w \otimes v) = ah^{-1} \otimes (hw \otimes hv)$, and the latter is mapped to $(ah^{-1} \otimes hw) \otimes ah^{-1}hv = (a \otimes w) \otimes av$, so η is well-defined. Let $g \in G$. Then $\eta(ga \otimes (w \otimes v)) = (ga \otimes w) \otimes gav = g \cdot (a \otimes w) \otimes av$, so η is a G-map.

In the other direction, define $\mu : \mathrm{Ind}_H^G(W) \otimes V \to \mathrm{Ind}_H^G(W \otimes \mathrm{Res}_H^G(V))$ by $\mu((a \otimes w) \otimes v) = a \otimes (w \otimes a^{-1}v)$. Clearly the two maps are inverse to each other, but we still must check that this is well-defined for the $\mathbb{C}H$ tensor product, and that this is a G-map. For $h \in H$, we have $(a \otimes w) \otimes v = (ah^{-1} \otimes hw) \otimes v$, and the latter expression is mapped to $ah^{-1} \otimes (hw \otimes (ah^{-1})^{-1}v) = ah^{-1} \otimes (hw \otimes ha^{-1}v) = a \otimes (w \otimes a^{-1}v)$. Thus μ is well-defined. Next, for $g \in G$, we have $\mu((ga \otimes w) \otimes gv) = ga \otimes (w \otimes (ga)^{-1}gv) = ga \otimes (w \otimes a^{-1}v)$, so μ is a G-map. Thus the two representations are isomorphic. □

(d) Show that $\mathrm{Ind}_H^G(\gamma \mathrm{Res}_H^G(\delta)) = \mathrm{Ind}_H^G(\gamma)\delta$.

Proof. First note that the Ind_H^G that has been defined on class functions acts on characters to give the character of the induced representation. That is, if $W = \mathrm{Ind}_H^G(V)$, then $\chi_W = \mathrm{Ind}_H^G(\chi_V)$ (we showed this in class). Thus, when γ is the character of an H representation W and δ is the character of a G representation V, the equation is exactly the equation between the characters of the representations $\mathrm{Ind}_H^G(W \otimes \mathrm{Res}_H^G(V))$ and $\mathrm{Ind}_H^G(W) \otimes V$ respectively. Then, from part (c), the equation holds.

In general, γ and δ can be expressed as a linear combination of characters. Thus, it suffices to show that the expressions on the left and right hand side of the equation are linear in γ and δ. Certainly, restriction of a function is linear, as is multiplying functions together. The "nontrivial" step is to show that Ind_H^G is linear. But this is also clear from the definition, as the sum defining $\mathrm{Ind}_H^G(\gamma)$ does not depend on γ, and sums are linear. Thus, we are done. □

2 Problem 4

Let G be a group with subgroup H (both possibly infinite) and let (W, ρ) be a representation of H over the field F. The coinduced representation of W is defined by

$$\mathrm{Coind}_H^G(W) = \{f : G \to W \mid f(gh^{-1}) = \rho(h)f(g) \text{ for all } h \in H, g \in G\},$$

with the G action given by $(x \cdot f)(g) = f(x^{-1}g)$.

(a) Show that $\mathrm{Coind}_H^G(W)$ is a representation isomorphic to $\mathrm{Hom}_H(FG, W)$ where H acts on FG on the right and the G action is $(g \cdot f)(u) = f(g^{-1}u)$.

Proof. It is a representation because $(xy \cdot f)(g) = f((xy)^{-1}g) = f(y^{-1}x^{-1}g) = (y \cdot f)(x^{-1}g) = (x \cdot (y \cdot f))(g)$ and $(e \cdot f)(g) = f(e^{-1}g) = f(g)$. The isomorphism with $\mathrm{Hom}_H(FG, W)$ is as follows: any $f \in \mathrm{Coind}_H^G(W)$ extends linearly to a map $FG \to W$, which is an H map because $f(h \cdot g) = f(gh^{-1}) = \rho(h)f(g) = h \cdot (f(g))$. Thus, we have a map in the forward direction. Conversely, a map $FG \to W$ always restricts to a map $G \to W$, and again, the H map condition is exactly the condition on the maps in $\mathrm{Coind}_H^G(W)$. Furthermore, the G-action on $\mathrm{Hom}_H(FG, W)$ is the same action on $\mathrm{Coind}_H^G(W)$, so the two representations are isomorphic. □

(b) Show that Coind_H^G is the right adjoint of Res_H^G.

Proof. We want to show that $\mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ is naturally isomorphic to $\mathrm{Hom}_G(V, \mathrm{Coind}_H^G W)$. For an H-map $f : V \to W$ and $v \in V$, let $\tilde{f}(v)(g) = f(g^{-1}v)$. Note that for $h \in H$ we have $\tilde{f}(v)(h \cdot g) = \tilde{f}(v)(gh^{-1}) = f(hg^{-1}v) = hf(g^{-1}v) = h\tilde{f}(v)(g)$, so that $\tilde{f}(v)$ is an H-map. Furthermore, for $x \in G$, $\tilde{f}(xv)(g) = f(g^{-1}xv) = \tilde{f}(v)(x^{-1}g) = (x \cdot \tilde{f}(v))(g)$, so that $\tilde{f}$ is a G-map. The map $f \mapsto \tilde{f}$ is then a map from $\mathrm{Hom}_H(\mathrm{Res}_H^G V, W)$ to $\mathrm{Hom}_G(V, \mathrm{Coind}_H^G W)$. As an inverse, we take $f' \in \mathrm{Hom}_G(V, \mathrm{Coind}_H^G W)$ to the map $\overline{f}'$ which sends $v \in V$ to $f'(v)(1)$. These are inverses because $\tilde{f}(v)(1) = f(1^{-1}v) = f(v)$ and $\overline{f}'(g^{-1}v) = f'(g^{-1}v)(1) = (g^{-1} \cdot f(v))(1) = f(v)(g)$. $\overline{f}'$ is an H-map because $\overline{f}'(hv) = f'(hv)(1) = (h \cdot f'(v))(1) = f'(v)(h^{-1}) = h \cdot (f(v)(1)) = h \cdot \overline{f}'(v)$. Thus we are done. □

(c) Show that $\mathrm{Ind}_H^G(W)$ is isomorphic to the subspace of $\mathrm{Coind}_H^G(W)$ consisting of the functions supported on finitely many left H-cosets. In particular, show that $\mathrm{Coind}_H^G(W) \cong \mathrm{Ind}_H^G(W)$ if H is of finite index in G. Can they be isomorphic if the index is infinite?

Proof. For $g \in G, w \in W$, send $g \otimes w \in \mathrm{Ind}_H^G(W)$ to the function $f_{g,w}$ defined by

$$f_{g,w}(x) = \begin{cases} (x^{-1}g)w & \text{if } x \in gH \\ 0 & \text{if } x \notin gH \end{cases}$$

First, this map is well-defined because $g \otimes w = gh^{-1} \otimes hw$ is sent to a map supported on $gh^{-1}H = gH$, sending $x \in gH$ to $x^{-1}gh^{-1}hw = x^{-1}gw$, so it is indeed $f_{g,w}$. Next, we show $f_{g,w}$ is compatible with the H action. This is $f_{g,w}(xh^{-1}) = hx^{-1}gw = h(f_{g,w}(x))$ for $x \in gH$, since $xh^{-1} \in gH$ also. For $x \notin gH$, we have $xh^{-1} \notin gH$, so both sides of the equation are 0. Next, we must show that the assignment $g \otimes w \mapsto f_{g,w}$ is a G-map. Indeed, for $y \in G$, we have $f_{yg,w}$ is supported on ygH, and for $x \in ygH$, $f_{yg,w}(x) = x^{-1}ygw$. But $x \in ygH$ if and only if $y^{-1}x \in gH$, in which case $f_{g,w}(y^{-1}x) = x^{-1}ygw$. Furthermore, $f_{g,w}(y^{-1}x) = (y \cdot f_{g,w})(x)$, so the assignment is a G-map.

We extend by linearity to a map defined on all of Ind. Notice that the sums involved in FG and $FG \otimes_{FH} W$ are finite, so that the image of Ind will consist of (a priori not necessarily all) functions supported on finitely many cosets of H.

If $f \in \text{Coind}_H^G W$ supported on finitely many cosets, then pick representatives $g_1, ..., g_n$ for the distinct(!) cosets it is supported on. Then, for $x \in g_iH$, write $x = g_ih$, so that $f(x) = h^{-1}f(g_i)$. In particular, f is determined by its values on the g_i. Let $w_i = f(g_i)$. Then, for $x = g_ih \in g_iH$, we have $f_{g_i,w_i}(x) = x^{-1}g_iw_i = h^{-1}f(g_i) = f(g_ih) = f(x)$. Then the element $\sum_{i=1}^n g_i \otimes w_i$ in $\text{Ind}_H^G W$ is mapped to f in $\text{Coind}_H^G W$. This construction is also independent of the choice of representatives: let $g_i' = g_ih_i$ for $h_i \in H$. Then

$$g_i'f(g_i') = g_ihf(g_ih) = g_if(g_ihh^{-1}) = g_if(g_i),$$

so that f_{g_i,w_i} is independent of the choice of representative for g_iH. Thus we get a map from the functions supported on finitely many cosets to Ind, showing that the two are indeed isomorphic.
If H is of finite index, then there are finitely many H-cosets, so all functions in $\text{Coind}_H^G(W)$ are necessarily supported on finitely many left H-cosets. Thus $\text{Coind}_H^G(W) \cong \text{Ind}_H^G(W)$ in this case.

If the index is infinite, you can construct a function which is not supported on finitely many cosets by choosing representatives g_i for all the cosets and setting $f(g_i) = w$ for some fixed non-zero element of W. The rest of the function is determined by the H-map property. So, Ind and Coind cannot be isomorphic, unless $W = 0$. $\square$

3 Problem 6

Let $G = SL_2(\mathbb{F}_p)$ where p is a prime.
(a) Show that G acts transitively on $X = \mathbb{F}_p^2 - \{(0,0)\}$ with the stabilizer of $(1,0)$ consisting of the matrices of the form $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$. What is $|G|$?

Proof. To show transitivity, it suffices to show that the orbit of $(1,0)$ is all of X. Let $(x,y) \in X$, and first assume that $x \neq 0$. Then $\begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix} \in SL_2(\mathbb{F}_p)$ and

$$\begin{pmatrix} x & 0 \\ y & x^{-1} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

If $x = 0$, then $y \neq 0$. Then $\begin{pmatrix} 0 & -y^{-1} \\ y & 0 \end{pmatrix} \in SL_2(\mathbb{F}_p)$ and

$$\begin{pmatrix} 0 & -y^{-1} \\ y & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix}.$$

Thus the action is transitive.

Suppose that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ fixes $(1,0)$. This forces the first column of the matrix to be $(1,0)$, i.e. $a = 1$ and $c = 0$. The determinant of such a matrix is d, so the condition of SL_2 implies $d = 1$. The only free parameter is the top-right entry, which can be anything in $\mathbb{F}_p$. In particular, the stabilizer of $(1,0)$ has p elements.

Finally, we have

$$|G| = |X| \cdot |\mathrm{Stab}((1,0))| = (p^2 - 1) \cdot p = p^3 - p.$$

by the orbit-stabilizer theorem. □

(b) Let B be the subgroup of upper triangular matrices. If α is a one-dimensional representation of $\mathbb{F}_p^*$, show that $\phi_\alpha \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \alpha(a)$ is a representation of B.

Proof. For $M_1, M_2 \in B$, we have

$$M_1 M_2 = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} w & x \\ 0 & z \end{pmatrix} = \begin{pmatrix} aw & ax + bz \\ 0 & dz \end{pmatrix},$$

so that $\phi_\alpha(M_1 M_2) = \alpha(aw)$. Since α itself is a representation, $\alpha(aw) = \alpha(a)\alpha(w) = \phi_\alpha(M_1)\phi_\alpha(M_2)$. This makes ϕ_α a one-dimensional representation of B. □

(c) Show that $\mathrm{Ind}_B^G(\phi_\alpha)$ is irreducible if and only if $\alpha^2 \neq 1$.

Proof. Let $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We want to show that $G = B \sqcup BsB$. First note that $B \cap BsB = \varnothing$, since

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} c & d \\ 0 & c^{-1} \end{pmatrix} = \begin{pmatrix} -bc & ac^{-1} - bd \\ -a^{-1}c & -a^{-1}d \end{pmatrix}$$

has a non-zero bottom-left entry. Now suppose $\begin{pmatrix} w & x \\ y & z \end{pmatrix} \in G$ with $y \neq 0$. Then

$$\begin{pmatrix} -1 & -wy^{-1} \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} y & z \\ 0 & y^{-1} \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix},$$

where the condition $wz - xy = 1$ is used. Thus $G = B \sqcup BsB$. Then $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and s are a complete set of double-coset representatives.

Now let $B_s = sBs^{-1} \cap B$. An element of sBs^{-1} is of the form

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ -b & a \end{pmatrix}.$$

Thus B_s consists of the diagonal matrices in G, which as a group is isomorphic to $\mathbb{F}_p^\times$ (and by a I will mean $\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$). In particular, B_s is a finite abelian group, so the irreps are one-dimensional and form a group under multiplication. Notice that ϕ_α restricts to α on B_s. For the conjugate representation, notice that

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}.$$

Then $\phi_\alpha^s(a) = \alpha(a^{-1})$. Thus,

$$\langle \phi_\alpha, \phi_\alpha^s \rangle = \frac{1}{p-1} \sum_{a \in \mathbb{F}_p^\times} \phi_\alpha(a) \phi_\alpha^s(a^{-1}) = \frac{1}{p-1} \sum_{a \in \mathbb{F}_p^\times} \alpha^2(a) = \langle \alpha^2, 1 \rangle.$$

Again, since the irreps form a group, α^2 is irreducible. Thus $\langle \alpha^2, 1 \rangle = 0$ iff $\alpha^2 \neq 1$. By Mackey's irreducibility criterion, we are done. □