MATH 7250 Homework 2

Andrea Bourque

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1 Problem 1

Let X be a finite set on which the group G acts, and let V be the corresponding complex permutation representation. Let χ be the character of this representation.

(a) Show that $\chi(g)$ is the number of elements of X fixed by g.

Proof. As a matrix in the natural basis for V, g is a permutation matrix, i.e. a matrix with exactly $|X|$ 1's and 0's everywhere else. 1's along the diagonal indicate a fixed point. The trace of the matrix is the sum of the 1's along the diagonal, which is equal to the number of fixed points. □

(b) Let c be the number of orbits of X. Show that V contains the trivial representation c times by finding an explicit basis for the trivial isotypic component. Conclude that there is a unique subrepresentation W such that $V = \mathbb{C}^c \oplus W$. Deduce that $\langle \chi, 1 \rangle = c$ and $\langle \chi_W, 1 \rangle = 0$.

Proof. If $\{x_1^j, ..., x_n^j\}$ is an orbit, then $\sum_i x_i^j$ is a fixed point, i.e. G acts trivially on $\mathbb{C}(\sum_i x_i^j)$. Thus there are at least c copies of the trivial representation occurring in V. Suppose v is also fixed by G. Write $v = \sum_{i=1}^n \sum_{j=1}^c a_i^j x_i^j$. Since the x_i^j form a basis for V, we can compare the coefficients of v and gv for all $g \in G$ to see that a_i^j is independent of the index i and we can call it a^j. Thus $v = \sum_{j=1}^c a^j \sum_{i=1}^n x_i^j$, implying that the trivial isotypic component is spanned by the c elements $\sum_i x_i^j$. Thus there are exactly c copies of the trivial representation occuring in V.

By the canonical decomposition of a representation into isotypic components, W exists and is unique, and is given by the direct sum of the non-trivial isotypic components.

Finally, we have that $\chi = c \cdot 1 + \chi_W$. Since χ_W doesn't contain any trivial subrepresentations by hypothesis (it is the complement to the trivial isotypic component), χ_W decomposes into a linear combination of irreducible characters excluding the trivial one. By orthogonality of irreducible characters, $\langle \chi_W, 1 \rangle = 0$. Then $\langle \chi, 1 \rangle = \langle c \cdot 1, 1 \rangle + \langle \chi_W, 1 \rangle = c + 0 = c$. □

(c) Assume X is transitive. Find the subrepresentation W explicitly.

Proof. Transitivity implies $c = 1$. In particular, the trivial isotypic component is spanned by $\sum_{x \in X} x$. Then W is the subspace of vectors $\sum_x a_x x$ such that $\sum_x a_x = 0$. □

(d) Let $G = S_n$ act naturally on $X = \{1, ..., n\}$. What is W?

Proof. The G-action here is transitive, so we can apply part c. In particular, W is the standard representation of S_n. □

2 Problem 4

If G is any topological group, let $\hat{G}$ denote the set of unitarizable continuous irreducible representations of G. (For finite groups, this is just all irreps).

(a) Show that the set of degree one representations of G is a group under pointwise multiplication. Conclude that $\hat{G}$ is a group for G finite abelian.

Proof. A degree one representation is just a group homomorphism $G \to \mathbb{C}^\times$. If we have two such representations ρ_1, ρ_2, then $\rho(g) = (\rho_1 \cdot \rho_2)(g) = \rho_1(g)\rho_2(g)$ is a group homomorphism since $\rho(gh) = \rho_1(gh)\rho_2(gh) = \rho_1(g)\rho_1(h)\rho_2(g)\rho_2(h) = \rho_1(g)\rho_2(g)\rho_1(h)\rho_2(h) = \rho(g)\rho(h)$. Note that we have used the abelian group structure on $\mathbb{C}^\times$. Thus we can multiply two representations and get another representation. The identity is the trivial representation, $e(g) = 1$ for all g. Inverses exist since $\rho(g)$ is always a nonzero complex number. Associativity follows from associativity of complex number multiplication.

When G is finite abelian, all the irreps are degree one, so $\hat{G}$, the set of all irreps, is a group by the previous work. □

(b) Show that $\hat{C}_n \cong C_n$. Is this isomorphism canonical?

Proof. A group homomorphism from C_n is always determined by where the generator is sent. Furthermore, it must be sent to an element with order dividing n. Thus $\hat{C}_n$ corresponds to the set of nth roots of unity in $\mathbb{C}$, which form the order n cyclic group under multiplication. The isomorphism is not canonical, because you can choose the generator of $\hat{C}_n$ to be any representation where the image of any generator of C_n is any primitive nth root of unity; there are two choices involved. □

(c) Show that $\hat{G} \cong G$ for any finite abelian group.

Proof. Any finite abelian group is a direct sum of cyclic groups, so from part (b) we just need to show that $\widehat{G \oplus H} \cong \hat{G} \oplus \hat{H}$. We can associate $\alpha \in \hat{G}, \beta \in \hat{H}$ to $\gamma \in \widehat{G \oplus H}$ via $\gamma(g,h) = \alpha(g)\beta(h)$. We can reverse the process by setting $\alpha(g) := \gamma(g,e)$, similarly for H. These are clearly homomorphisms and the constructions are inverse to each other, so we are done. □

(d) For G finite abelian, define $\text{ev} : G \to \hat{\hat{G}}$ via $x \mapsto \text{ev}_x$. Show that this map is an isomorphism.

Proof. Note that $\text{ev}_{xy}(\alpha) = \alpha(xy) = \alpha(x)\alpha(y) = (\text{ev}_x \cdot \text{ev}_y)(\alpha)$, so ev is a homomorphism. By applying part (c) twice, the two groups have the same order, so it suffices to show ev is injective. If $\text{ev}_x(\alpha) = 1$ for all α, i.e. $\alpha(x) = 1$ for all α, then the column corresponding to the conjugacy class of x for the character table of G has all 1's (since each irrep is one dimensional). This would imply (since we are in characteristic 0) that the column is not orthogonal to the column corresponding to e. Then it must be that $x = e$, so ev is injective. □

(e) If $\alpha \in \hat{G}$, show that the α-isotypical part of the regular representation is spanned by $\sum_g \overline{\alpha}(g)g$.

Proof. (I am pretty sure we assume G is finite abelian here, as this says all irreps are one-dimensional.) Notice that the element $\sum_g \overline{\alpha}(g)g$ as a map on the vector space is $\sum_g \overline{\alpha}(g)\alpha(g) = \sum_g |\alpha(g)|^2 = |G|$ ($|\alpha(g)|^2 = 1$ since $\alpha(g)$ is a root of unity). On the other hand, for any $\beta \in \hat{G}$ which is not α, $\sum_g \overline{\alpha}(g)g$ acts on the corresponding vector space as $\sum_g \overline{\alpha}(g)\beta(g) = 0$ by orthogonality of characters. This element is then a non-zero element of the α isotypic component. Since isotypic components in the regular representation have the square of the dimension of the corresponding irrep, and the irreps of G have dimension one, the isotypic component has dimension one. Thus, the element spans the α isotypic component. □

3 Problem 6

Assume that char $F \nmid |G|$. Let $V_1, ..., V_r$ be the irreps of G. Let W be a representation and denote the isotypic component corresponding to V_i by W_i.

(a) View $\text{Hom}_G(V_i, W)$ as a representation via the trivial action. Show that the map $\eta : \bigoplus_{i=1}^r \text{Hom}_G(V_i, W) \otimes V_i \to W$ given by $(\alpha_i \otimes v_i)_{i=1}^r \mapsto \sum_{i=1}^r \alpha_i(v_i)$ is a G-isomorphism.

Proof. η is a G-map, because $g(\alpha_i \otimes v_i)_{i=1}^r = (g\alpha_i \otimes gv_i)_{i=1}^r = (\alpha_i \otimes gv_i)_{i=1}^r$ is mapped to $\sum_{i=1}^r \alpha_i(gv_i) = \sum_{i=1}^r g(\alpha_i(v_i)) = g \sum_{i=1}^r \alpha_i(v_i)$, where the second to last inequality is due to the fact that the α_i are themselves G-maps.

Recall that $\dim(\text{Hom}_G(V_i, W))$ is the multiplicity of V_i in W by Schur's lemma; call it m_i. Note too that $\dim W_i = m_i \dim V_i$. Then

$$\begin{aligned}\dim\left(\bigoplus_{i=1}^r \text{Hom}_G(V_i, W) \otimes V_i\right) &= \sum_{i=1}^r \dim(\text{Hom}_G(V_i, W)) \dim V_i \\ &= \sum_{i=1}^r m_i \dim V_i = \sum_{i=1}^r \dim W_i = \dim W.\end{aligned}$$

Thus, it suffices to show that η has trivial kernel. First, note that $\alpha_i(v_i) \in W_i \subset W$ by Schur's lemma. In particular, considering the projection maps $\pi_j^{k_j}$ from the W_j to the copies of V_j in W, $\pi_j^{k_j}\alpha_i : V_i \to V_j$ must be the 0 map when $i \neq j$. Since $W = \bigoplus_{i=1}^r$, $\sum_{i=1}^r \alpha_i(v_i) = 0$ if and only if $\alpha_i(v_i) = 0$ for each i. Since α_i is a g map, $\alpha_i(v_i) = 0$ means $\alpha_i(gv_i) = 0$ for all g. Since V_i is irreducible, $\text{span}_F\{gv_i \mid g \in G\}$ is 0 if $v_i = 0$ or V if $v_i \neq 0$. In the latter case, $\alpha_i = 0$. Thus, in either case, $\alpha_i \otimes v_i = 0$. Thus η has trivial kernel. $\square$

(b) Show that $\text{Hom}_G(V_i, FG)$ is a representation of G via the action $(g \cdot \alpha)(v) = \alpha(v)g^{-1}$ and that this representation is isomorphic to V_i^*.

Proof. First, $(e \cdot \alpha)(v) = \alpha(v)e^{-1} = \alpha(v)$, so $e \cdot \alpha = \alpha$. Next, $((gh) \cdot \alpha)(v) = \alpha(v)(gh)^{-1} = \alpha(v)h^{-1}g^{-1} = (g \cdot (h \cdot \alpha))(v)$, so $(gh) \cdot \alpha = g \cdot (h \cdot \alpha)$. Furthermore, g is linear since it just acts as multiplication of an algebra element, and algebra multiplication is linear. This shows we indeed have a representation.

Note that $\dim(\text{Hom}_G(V_i, FG))$ is the multiplicity of V_i in FG, which we have seen is $\dim V_i$, which is equal to $\dim(V_i^*)$. Therefore, it suffices to construct an injective linear G-map $\phi : V_i^* \to \text{Hom}_G(V_i, FG)$. To that end, define $(\phi(\alpha))(v) = \sum_{g \in G} \alpha(g^{-1}v)g$. First, we must check that $\phi(\alpha)$ is a G-map. Indeed,

$$\begin{aligned}(\phi(\alpha))(hv) &= \sum_{g\in G} \alpha(g^{-1}hv)g = \sum_{g\in G} \alpha((h^{-1}g)^{-1}v)hh^{-1}g \\ &= h\sum_{g'\in G} \alpha(g'^{-1}v)g' = h((\phi(\alpha))(v)).\end{aligned}$$

Next, we must show ϕ is a G-map. Recall the action on V_i^* is $(g\cdot\alpha)(v) = \alpha(g^{-1}v)$. Then

$$\begin{aligned}(\phi(h\cdot\alpha))(v) &= \sum_{g\in G}(h\cdot\alpha)(g^{-1}v)g = \sum_{g\in G}\alpha(h^{-1}g^{-1}v)g \\ = \sum_{g\in G}\alpha((gh)^{-1}v)ghh^{-1} &= \sum_{g'\in G}\alpha(g'^{-1}v)g'h^{-1} = (\phi(\alpha))(v)h^{-1} \\ &= (h\cdot(\phi(\alpha)))(v).\end{aligned}$$

ϕ is obviously linear by construction (read: I'm lazy). The last thing to check is that ϕ is injective. If $\phi(\alpha)$ is the 0 map, then $\sum_{g\in G}\alpha(g^{-1}v)g = 0$ for all $v \in V_i$. This sum is 0 if and only if each coefficient is 0, since the g are linearly independent. In particular, we look at the e coefficient of this sum, which is just $\alpha(v)$; $\alpha(v) = 0$ for all v, so $\alpha = 0$. Thus we are done. □

(c) Show that the canonical algebra isomorphism $FG \to \bigoplus_{i=1}^r \operatorname{End}(V_i)$ is an isomorphism of $G\times G$-modules which gives the decomposition of FG into irreducible $G\times G$-modules. The $G\times G$-module structures on FG and $\operatorname{End}(V_i)$ are given by $(g,h)\cdot\gamma = g\gamma h^{-1}$ and $((g,h)\cdot\beta)(w) = g(\beta(h^{-1}w))$.

Proof. Note that the $G\times G$-module structure on $\operatorname{End}(V_i)$ really is just $(g,h)\cdot\beta = g\circ\beta\circ h^{-1}$, where the group elements on the right hand side are viewed as linear maps via the representation V_i. This is essentially the same as the $G\times G$ module structure on FG. Since the map $FG\to\operatorname{End}(V_i)$ takes an element to the map it induces by interpreting the group elements as linear maps, these two module structures are obviously compatible with the algebra isomorphism $FG \to \bigoplus_{i=1}^r \operatorname{End}(V_i)$, making it an isomorphism of $G\times G$-modules.

Now we must check that $\operatorname{End}(V_i)$ is irreducible. Recall that $G\times H$ irreps come from tensoring G irreps and H irreps. As G-modules, $\operatorname{End}(V_i)\cong V_i^*\otimes V_i$. The dual of an irrep is an irrep, so $\operatorname{End}(V_i)$ is a $G\times G$ irrep from the previous result. □