MATH 7250 Homework 1

Andrea Bourque

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1 Problem 1

Let F be an algebraically closed field.

(a) Show that if A is a finitely generated commutative F-algebra, then every simple A-module is 1-dimensional.

Proof. It is a standard fact that every simple module for a commutative ring is isomorphic to a quotient of the ring by a maximal ideal. Thus let M be a simple A-module which is isomorphic to $A/\mathfrak{m}$ for a maximal ideal $\mathfrak{m}$. Of course, $A/\mathfrak{m}$ can be considered as a field. Since A is finitely generated over F, so is $A/\mathfrak{m}$ (as an F-algebra). Zariski's lemma then says that $A/\mathfrak{m}$ is a finite field extension of F. Since F is algebraically closed, this means that $A/\mathfrak{m}$ is F, which means M is one-dimensional. □

(b) For prime p, find the irreducible representations of the cyclic group Z_p over F.

Proof. Since Z_p is abelian and F is algebraically closed, the irreps are one-dimensional. Furthermore, a representation of a cyclic group is determined by its image at a generator. The image ρ must satisfy $\rho^p = 1$. Suppose char $F \neq p$. Then there are p distinct pth roots of unity, each one giving an irrep of Z_p. If char $F = p$, then there is only the trivial representation, since $(x-1)^p = x^p - 1$ shows that 1 is the only pth root of unity. □

(c) Find the irreducible representations of Z_p over $\mathbb{R}$.

Proof. Fix a generator of Z_p. Let V be an irrep, and let $A \in GL_{\mathbb{R}}(V)$ be the image of the generator. Then $A^p = \text{id}$. A has a complex eigenpair (λ, v). If it happens that $\lambda \in \mathbb{R}$, then $\mathbb{R}v$ is an invariant subspace. Since V is irreducible, $V = \mathbb{R}v$. If $\lambda \notin \mathbb{R}$, then the real and imaginary parts of v span an invariant subspace. Explicitly, $A\text{Re}(v) = \text{Re}(\lambda)\text{Re}(v) - \text{Im}(\lambda)\text{Im}(v)$ and $A\text{Im}(v) = \text{Im}(\lambda)\text{Re}(v) + \text{Re}(\lambda)\text{Im}(v)$. Again, since V is irreducible, V must be spanned by $\text{Re}(v), \text{Im}(v)$. Notice that we could also choose the eigenpair $(\overline{\lambda}, \overline{v})$ and get the same conclusion. That is to say, the irrep is not classified by its eigenvalue, but by the unordered pair $\{\lambda, \overline{\lambda}\}$.

Now, an eigenvalue of A must satisfy $\lambda^p = 1$. In the case $p = 2$, we have two possible real eigenvalues ± 1, each giving a 1-dimensional irrep. For $p > 2$, The only real eigenvalue is 1, which gives the 1-dimensional trivial irrep. Otherwise, $\lambda = \exp(2\pi i k/p)$ for $k \in \{1, ..., p-1\}$. Since $\overline{\lambda}$ is also an eigenvalue, we get $(p-1)/2$ 2-dimensional irreps this way. They are all non-isomorphic because the trace of A is $\lambda + \overline{\lambda} = 2\cos(2\pi k/p)$. □

2 Problem 4

Let V be a complex irreducible representation of a finite group G. Show that there is a unique G-invariant Hermitian inner product on V up to positive scalar.

Proof. First, a G-invariant Hermitian inner product exists by the averaging argument: We take any Hermitian inner product H (there is a canonical one assigned to any choice of basis), and then replace it by

$$H_G(v, w) = \frac{1}{|G|} \sum_{g \in G} H(gv, gw).$$

Let H and H' be two G-invariant Hermitian inner products on V. Consider their induced conjugate linear maps $\tilde{H} : v \mapsto H(v, -)$ and $\tilde{H}' : v \mapsto H'(v, -)$. Since V is finite dimensional, $\tilde{H}$ is invertible. Then consider $\tilde{H}^{-1} \circ \tilde{H}' : V \to V$. If this is a G-map, then it will be a scalar by Schur's lemma. Then $\tilde{H}' = \tilde{H} \circ \lambda$, meaning $H'(v, w) = H(\lambda v, w) = \lambda H(v, w)$ for all v, w. Setting $v = w \neq 0$ implies that λ must be a positive real number.

We now show $\tilde{H}^{-1} \circ \tilde{H}'$ is a G-map. For any H-orthonormal basis e_i (one exists by Gram-Schmidt process), $\tilde{H}^{-1}(\alpha) = \sum_i \alpha(e_i)e_i$. Since H is G-invariant, for any g, ge_i is an H-orthonormal basis, since $H(ge_i, ge_j) = H(e_i, e_j) = \delta_{ij}$, and an equation of linear dependence would imply g is not invertible. Hence $\tilde{H}^{-1}(\alpha) = \sum_i \alpha(ge_i)ge_i$. Finally, we have

$$\begin{aligned} (\tilde{H}^{-1} \circ \tilde{H}')(gv) &= \sum_i H'(gv, ge_i)ge_i = \sum_i H'(v, e_i)ge_i \\ &= g \sum_i H'(v, e_i)e_i = g(\tilde{H}^{-1} \circ \tilde{H}')(v). \end{aligned}$$

This concludes the proof. □

3 Problem 6

Let F be any field.

(a) Show that $P : \mathbf{Grp} \to \mathbf{F\text{-}Alg}$ as defined below is a functor:

- $P(G) = FG$
- $P(f : G \to H)$ is the linear extension of f to a map $FG \to FH$.

Proof. We must check that P preserves identity morphisms and composition. For $\mathrm{id}_G : G \to G$, we have $P(\mathrm{id}_G)(\sum a_g g) = \sum a_g \mathrm{id}_G(g) = \sum a_g g$, so $P(\mathrm{id}_g) = \mathrm{id}_{FG}$.

Now let $f : G \to H$ and $h : H \to K$. We have $P(h \circ f)(\sum a_g g) = \sum a_g h(f(g))$, whereas $(P(h) \circ P(f))(\sum a_g g) = P(h)(\sum a_g f(g)) = \sum a_g h(f(g))$, so $P(h \circ f) = P(h) \circ P(f)$. □

(b) Given an algebra morphism $f : A \to B$, define $f^\times : A^\times \to B^\times$ so that $^\times$ is a functor $\mathbf{F\text{-}Alg} \to \mathbf{Grp}$.

Proof. If $a \in A^\times$, then $f(a) \in B^\times$ since it has an inverse $f(a^{-1})$. Thus we take $f^\times : A^\times \to B^\times$ to be the restriction of f to $A^\times$. Restriction preserves identity and composition, so $^\times$ is a functor. □

(c) Show that P is left adjoint to $^\times$.

Proof. Given a group G and an F-algebra A, we construct $\Phi(G, A)$ as follows. For a group morphism $f : G \to A^\times$, let $\Phi(G, A)(f) : P(G) \to A$ send $\sum a_g g$ to $\sum a_g f(g)$. In the opposite direction, suppose we are given an F-algebra morphism $f : FG \to A$. Consider $g \in G$ as an element $1g \in FG$; it is a unit, so $f(1g)$ is a unit as well. Thus, we get a group morphism $f_G : G \to A^\times$ by restricting f to the elements $1g$. Since knowing the action on the "pure" elements $1g \in FG$ is enough to determine a morphism out of FG, we see that these constructions are inverse to each other, establishing that $\Phi(G, A)$ is a bijection.

Next, we show that $\Phi(G, A)$ is natural in G and A. Let $\alpha : G' \to G$ be a group morphism and let $\beta : A \to A'$ be an F-algebra morphism. Given $f : G \to A^\times$, $\Phi(G', A')(\beta^\times \circ f \circ \alpha)$ sends $\sum a_{g'} g' \in FG'$ to $\sum a_{g'} \beta^\times(f(\alpha(g')))$. Recall that $\beta^\times$ is just a restriction of β, so that $\sum a_{g'} \beta^\times(f(\alpha(g'))) = \sum a_{g'} \beta(f(\alpha(g'))) = \beta(\sum a_{g'} f(\alpha(g')))$. Now, $\sum a_{g'} f(\alpha(g')) = \Phi(G, A)(f)(\sum a_{g'} \alpha(g'))$, and $\sum a_{g'} \alpha(g') = P(\alpha)(\sum a_{g'} g')$, so $\Phi(G', A')(\beta^\times \circ f \circ \alpha) = \beta \circ \Phi(G, A)(f) \circ P(\alpha)$. □