MATH 7240 Homework 3

Andrea Bourque

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1 Problem 21.1

Let $R = k[x, y]$ and $I = \langle x, y \rangle$. Show that the Rees algebra $\text{Rees}_R(I)$ is isomorphic to the graded algebra $R[U, V]/\langle xV - yU \rangle$, where U, V have degree 1.

Proof. Write the Rees algebra as $\bigoplus_{n=0}^{\infty} I^n t^n$. The correspondence is that t "is the same as" $\frac{U}{x}$, which is also equal to $\frac{U}{y}$. For instance, in degree 1, an element of $R[U, V]/\langle xV - yU \rangle$ is

$$fU + gV = fU + g\frac{yU}{x} = (xf + yg)\frac{U}{x},$$

and $xf + yg \in \langle x, y \rangle$. In general, a degree n element of $R[U, V]/\langle xV - yU \rangle$ IS

$$\sum f_i U^i V^{n-i} = \sum f_i \frac{y^{n-i}}{x^{n-i}} U^n = \left(\sum f_i x^i y^{n-i}\right) \left(\frac{U}{x}\right)^n,$$

and $\sum f_i x^i y^{n-i} \in I^n$, so the isomorphism is clear. □

2 Problem 21.2

With the notation of Problem 21.1, the blow up $\mathrm{Bl}_R(I)$ is isomorphic to the closed subscheme of $\mathbb{P}^1_R$ defined by $xV - yU = 0$. This is covered by two open sets $U_1 = D(U), U_2 = D(V)$. Show that $U_1 = \mathrm{Spec}(k[x, \frac{V}{U}]), U_2 = \mathrm{Spec}(k[\frac{U}{V}, y])$, and the map $\pi : \mathrm{Bl}_R(I) \to \mathrm{Spec}(R)$ is given on these open sets by $(x, \frac{V}{U}) \to (x, x\frac{V}{U})$ and $(\frac{U}{V}, y) \to (y\frac{U}{V}, y)$.

Proof. By definition, U_1 is Spec of the degree 0 part of $R[U, V]/\langle xV - yU\rangle$ localized at U. Degree 0 elements of this localization are of the form f/U^n for f of degree n, say $f = \sum f_i U^i V^{n-i}$ for $f_i \in R$. Then $f/U^n = \sum f_i (V/U)^{n-i}$, so we have $R[V/U]$. Furthermore, since we are on the subscheme defined by $xV - yU = 0$, we have $y = xV/U$, so $R[V/U] = k[x, V/U]$ as desired.

The blowup map π is induced by the map $R \to \mathrm{Rees}_R(I) = R[U, V]/\langle xV - yU\rangle$ which simply takes a polynomial to the degree 0 part. The map $\pi : U_1 \to \mathrm{Spec}(R)$ is then naturally given by the map $R \to k[x, V/U]$ which maps y to xV/U.

A similar analysis gives the desired results for U_2. $\square$

3 Problem 22.1

If $f : Y \to X$ is a continuous map of Noetherian spaces, and if $E \subset X$ is constructible, then $f^{-1}(E)$ is constructible.

Proof. In Noetherian spaces, constructible sets are those sets which are finite unions of locally closed sets. In general, inverse images preserve finite unions and finite intersections. Since f is continuous, f^{-1} preserves closed and open sets. Thus the inverse image of a finite union of locally closed sets is the finite union of locally closed sets as desired. □

4 Problem 22.2

Let X be Noetherian, and let $Y \subset X$ be constructible. Show that the subsets of Y which are constructible in Y are exactly those subsets of Y which are constructible in X.

Proof. Write $Y = \bigcup_{i=1}^n U_i \cap V_i$, for open sets U_i and closed sets V_i. Let $E \subset Y$. First suppose $E = \bigcup_{j=1}^m A_j \cap B_j$ for A_j open in Y, B_j closed in Y. The open and closed subsets of Y are, by definition, intersections of Y with open and closed subsets of X, respectively. Thus write $A_j = A'_j \cap Y, B_j = B'_j \cap Y$ for A'_j, B'_j open and closed in X. Then

$$E = \bigcup_{j=1}^m A_j \cap B_j = \bigcup_{j=1}^m (A'_j \cap Y) \cap (B'_j \cap Y) = Y \cap \bigcup_{j=1}^m A'_j \cap B'_j.$$

Thus E is the intersection of constructible subsets of X, which is constructible. Now suppose $E = \bigcup_{j=1}^m P_j \cap Q_j$ for P_j, Q_j open and closed in X. Since $E \subset Y$, we have $P_j \cap Q_j \subset Y$ for each j. Then

$$P_i \cap Q_i = (P_i \cap Q_i) \cap Y = (P_i \cap Y) \cap (Q_i \cap Y)$$

is locally closed in Y, so E is constructible in Y. □

5 Problem 22.3

Let X be irreducible. Show that a set $E \subset X$ is nowhere dense if and only if E is not dense.

Proof. ($\rightarrow$) Nowhere dense means that $X - \overline{E}$ is dense. If E is dense, then we have $\overline{\varnothing} = X$, i.e. $X = \varnothing$. So E is not dense (assuming X is not empty).

($\leftarrow$) Suppose E is not dense. Note that $X = \overline{(X - \overline{E})} \cup \overline{E}$. Since X is irreducible and $\overline{E} \neq X$, we must have $\overline{X - \overline{E}} = X$, so X is nowhere dense. $\square$

6 Problem 22.4

Let X be a Noetherian irreducible space. Let $E \subset X$ be constructible. Show that E is dense if and only if E contains a non-empty open subset of X.

Proof. ($\rightarrow$) Suppose E is dense. Let $E = \bigcup_{i=1}^{n} U_i \cap V_i$, where U_i, V_i are open and closed in X respectively. Then

$$X = \overline{E} = \bigcup_{i=1}^{n} \overline{U_i \cap V_i} \subset \bigcup_{i=1}^{n} V_i.$$

Since X is irreducible, this means X is equal to one of V_i; in particular, all the V_i are either X or $\varnothing$. Assuming X is non-empty, we must have E is non-empty. Then for one of the V_i which is equal to X, the corresponding U_i must be non-empty (otherwise E would be the union of empty sets). Then $U_i \cap V_i = U_i$ is a non-empty open subset of E.

($\leftarrow$) Let U be a non-empty subset of E which is open in X. Since X is irreducible and $X = \overline{U} \cup (X - U)$, we have $\overline{U} = X$. Thus $\overline{E} = X$ also. ☐

7 Problem 22.5

Let X be a Noetherian topological space. Then $E \subset X$ is constructible if and only if, for every closed irreducible subset Y of X, either $E \cap Y$ is nowhere dense in Y, or $E \cap Y$ contains an open subset of Y.

Proof. ($\rightarrow$) Note that $E \cap Y$ is constructible in Y; a short proof is to apply Problem 22.1 to the inclusion map $i : Y \hookrightarrow X$, noting also that Y is Noetherian, since any subspace of a Noetherian space is Noetherian. Now, $E \cap Y$ is either dense or not dense. By Problems 22.3 and 22.4, we get that either $E \cap Y$ is nowhere dense or E contains a non-empty open subset of Y.

($\leftarrow$) We use Noetherian induction. Suppose that for all proper closed subsets $Y \subset X$, we have $E \cap Y$ is constructible (Note: Problem 22.2 ensures that it doesn't matter whether $E \cap Y$ is constructible in Y or in X). Note that this property is true when $Y = \varnothing$. Since X is Noetherian, it has a finite number of irreducible components. If X is not irreducible, then it is a finite union of proper closed (irreducible) subsets X_i, for which each $E \cap X_i$ is constructible by assumption. Then E is a finite union of constructible sets, which is constructible. If X is irreducible, then we are assuming that either E is nowhere dense in X or E contains an open subset of X. If E is nowhere dense, then from Problem 22.3 it is not dense, so $\overline{E}$ is a proper closed set, whence $E = E \cap \overline{E}$ is constructible by the induction hypothesis. If E contains a non-empty open subset U of X, then $(X - U) \cap E$ is constructible by the induction hypothesis. Then $E = ((X - U) \cap E) \cup U$ is the union of two constructible sets (opens are constructible), so E is constructible. Thus, in either case, E is constructible, so we are done by induction. □

8 Problem 23.1

Consider the morphism $f : \mathbb{P}^1 \to \mathbb{P}^3$ given by the invertible sheaf $\mathcal{L} = \mathcal{O}(4)$ on $\mathbb{P}^1$ (coordinates s, t) and the four sections s^4, s^3t, st^3, t^4. Determine the image of f as $V(J)$ for a homogenous ideal J in $k[w, x, y, z]$.

Proof. We use singular. Set up a ring with variables s, t, w, x, y, z, create the ideal generated by $w - s^4, x - s^3t, y - st^3, z - t^4$, and then use the eliminate function to get the generators in w, x, y, z. This gives $J = (wz - xy, xz^2 - y^3, wy^2 - x^2z, w^2y - x^3)$. □

9 Problem 27.1

Let $\alpha : k[x_1, x_2, x_3]/(x_1x_3 - x_2^2) \to k[x, y], (x_1, x_2, x_3) \mapsto (x^2, xy, y^2)$ induce the morphism $f : \mathbb{A}_k^2 = \mathrm{Spec}(k[x, y]) \to X = \mathrm{Spec}(k[x_1, x_2, x_3]/(x_1x_3 - x_2^2))$. Show that f is flat over every (a, b, c) with $ac = b^2 \neq 0$, but not over $(0, 0, 0)$.

Proof. The morphism f is finite, so in order to be flat, $f_*(\mathcal{O}_{\mathbb{A}_k^2})$ must be locally free; in other words, the stalks over points in X should have the same rank. For $p = (a, b, c) \in X$ with $ac = b^2$, the residue field is $k[x_1, x_2, x_3]/(x_1 - a, x_2 - b, x_3 - c)$. Then the stalk of $f_*(\mathcal{O}_{\mathbb{A}_k^2})$ at p is

$$\begin{aligned} k[x, y] \otimes_{k[x_1,x_2,x_3]/(x_1x_3-x_2^2)} & k[x_1, x_2, x_3]/(x_1 - a, x_2 - b, x_3 - c) \\ &= k[x, y]/(x^2 - a, xy - b, y^2 - c). \end{aligned}$$

When $(a, b, c) = (0, 0, 0)$, we have $k[x, y]/(x^2, xy, y^2)$, which is clearly 3-dimensional with basis $\{1, x, y\}$. If $ac = b^2 \neq 0$, then we have the non-trivial linear relationship $bx = x^2y = ay$, so that the spanning set $\{1, x, y\}$ is no longer independent. Thus the rank is lower on a generic point, and f cannot be flat. $\square$

10 Problem 27.2

Consider the family of space curves given by $\alpha_a(t) = (t^2 - 1, t^3 - t, at)$. Verify that the closure of the image of α_a is the subvariety $X = V(I) \subset \mathbb{A}^3_{k[a]}$, where

$$I = \langle a^2(x+1) - z^2, ax(x+1) - yz, xz - ay, y^2 - x^2(x+1)\rangle.$$

Show that $k[a, x, y, z]/I$ is torsion-free over $k[a]$, which shows that the map $X \to \mathbb{A}^1_{k[a]}$ is flat.

Proof. To compute I, we use Singular. Set up a ring with variables a, t, x, y, z and an ideal with generators $x - t^2 + 1, y - t^3 + t, z - at$, and then use the eliminate function to get rid of t. This gives the stated expression of I. To find that $k[a, x, y, z]/I$ is torsion-free over $k[a]$, we can use Singular again. Import control.lib and use the function findTorsion(A,I) where $A = k[a]$ as a module over $k[a, x, y, z]$. Then the CAS displays that the torsion is 0. $\square$

11 Problem 28.1

Locate the singular points of the following curves in $\mathbb{A}_k^2$ with char $k \neq 2$.

1. $x^2 = x^4 + y^4$.
2. $xy = x^6 + y^6$.
3. $x^3 = y^2 + x^4 + y^4$.
4. $x^2y + xy^2 = x^4 + y^4$.

Proof. We use the Jacobian criterion for each equation. For (1), we have the Jacobian is $(4x^3 - 2x, 4y^3)$. The second component is 0 iff $y = 0$. Putting $y = 0$ into the equation for the curve gives $x^2 = x^4$, whence $x \in \{-1, 0, 1\}$. Only $x = 0$ gives $4x^3 - 2x = 0$, so the only singularity of this curve is $(0, 0)$. Hartshorne calls this a tacnode.

For (2), we have the Jacobian is $(6x^5 - y, 6y^5 - x)$. If the Jacobian vanishes, then $6x^6 = xy = 6y^6$, but we know $xy = x^6 + y^6$, so $5x^6 = y^6$ and $5y^6 = x^6$. Combining these two equations gives $24x^6 = 0$ and $24y^6 = 0$, whence $(x, y) = (0, 0)$, assuming char $k \neq 3$. If char $k = 3$, the Jacobian is $(-y, -x)$, and the only singularity is still $(0, 0)$. Hartshorne calls this a node.

For (3), The Jacobian is $(4x^3 - 3x^2, 4y^3 + 2y)$. The first component vanishes when $x = 0$ or $x = 4/3$, if char $k \neq 3$. When $x = 0$, we have $y^2 + y^4 = 0$, so $y = 0$ or $y^2 = -1$. The former case makes the Jacobian vanish, while the latter makes the second component $-2y \neq 0$. Thus $(0, 0)$ is a singularity. If $x = 4/3$, then $y \neq 0$ from the curve equation. Then the Jacobian vanishing forces $y^2 = -1/2$, and we can check that $(4/3)^3 \neq -1/2 + (4/3)^4 + (-1/2)^2$. Thus in this case, the only singularity is $(0, 0)$. If char $k = 3$, then the Jacobian is $(x^3, y^3 - y)$. Then we must have $x = 0$, so $y = 0$ or $y^2 = -1$. As in the previous case, $y^2 = -1$ does not give a singularity, so the only singularity is $(0, 0)$. Hartshorne calls this a cusp.

For (4), the Jacobian is $(4x^3 - 2xy - y^2, 4y^3 - 2xy - x^2)$. Suppose the Jacobian vanishes. Multiply the two equations from this condition by x and y respectively, and add them together. This gives $x^2y + xy^2 = xy(x + y) = 0$. From the curve equation, $x = 0$ iff $y = 0$. If $x + y = 0$, then from the curve equation $2x^4 = 0$. This means $(0, 0)$ is the only singularity. Hartshorne calls this a triple point (note that the lowest degree term is 3, as opposed to the other equations). □

12 Problem 28.2

Locate the singular points of the following surfaces in $\mathbb{A}_k^3$ with char $k \neq 2$.

1. $xy^2 = z^2$.
2. $x^2 + y^2 = z^2$.
3. $xy + x^3 + y^3 = 0$.

Proof. For (1), the Jacobian is $(y^2, 2xy, -2z)$. Then a singular point must occur when $y = z = 0$, which automatically satisfies the curve equation and the vanishing of the second component of the Jacobian. In other words, $(x, 0, 0)$ is a singularity for all $x \in k$. Hartshorne calls this a pinch point.

For (2), the Jacobian is $(2x, 2y, -2z)$, so the only singularity is $(0, 0, 0)$. Hartshorne calls this a conical double point.

For (3), the Jacobian is $(3x^2 + y, x + 3y^2, 0)$. Setting the first and second components to 0, multiplying by x and y respectively, and then adding gives $xy = 0$. This $3x^3 = 3y^3 = 0$ from the Jacobian vanishing equations multiplied. If char $k \neq 3$, this implies $x = y = 0$, so the singularities are $(0, 0, z)$ for all $z \in k$. If char $k = 3$, the Jacobian is just $(y, x, 0)$, so clearly the only singularities are $(0, 0, z)$ for all $z \in k$. Hartshorne calls this a double line. □

13 Problem 33.1

Let X be a curve, and let $P \in X$ be a point. Show that there exists a nonconstant rational function $f \in K(X)$ which is regular everywhere but P.

Proof. Let $Q \in X$ be a point distinct from P. Consider $D = nP - Q$ for n a positive integer. If $n > g$, the genus of X, then by Riemann Roch, $\chi(\mathcal{L}(D)) = n - 1 + 1 - g > 0$. In particular, $\dim H^0(X, \mathcal{L}(D)) \geq 1$, which means there is a non-zero function $f \in K(X)$ with $div(f) + D \geq 0$. If f is constant, then $div(f) = 0$, but $D \not\geq 0$, so f is not constant. We know that $\nu_Q(f) - 1 \geq 0$, $\nu_P(f) + n \geq 0$, and $\nu_R(f) \geq 0$ for all points $R \in X$ distinct from P and Q. Thus $f(Q) = 0$, and since $\deg(div(f)) = 0$, we know f must have a pole somewhere. But f cannot have any poles on points that are not P or Q, and we just said it does not have a pole at Q, so it only has a pole at P. □

14 Problem 33.2

Let X be a curve, and let $P_1, ..., P_r \in X$ be points. Then there is a function $f \in K(X)$ which only has poles at the P_i.

Proof. From Problem 33.1, there are functions $f_1, ..., f_r \in K(X)$ such that f_i is regular everywhere but P_i. Then $f = f_1 + ... + f_r$ is regular everywhere but the P_i, and furthermore, since each f_j for $j \neq i$ is regular at P_i, we must have that f is not regular at P_i. In other words, there cannot be any cancellation of the singularities, since they occur at distinct points. $\square$

15 Problem 33.1

Let X be a curve of genus g over an algebraically closed field k. Show that there is a finite morphism $f : X \to \mathbb{P}^1$ of degree $\leq g+1$.

Proof. We invoke the following lemma:

Lemma. For a divisor C and a point P, we have $l(C)$ is either $l(C)$ or $l(C)+1$.

Proof of the Lemma. The rational functions satisfying $f + C \geq 0$ must also satisfy $f + C + P \geq 0$, so $\mathcal{L}(C)$ is a subsheaf of $\mathcal{L}(C+P)$. Furthermore, they are the same except near P. It follows that the quotient sheaf is a skyscraper sheaf at P with value k, denoted k_P. Then we have the exact sequence

$$0 \to \mathcal{L}(C) \to \mathcal{L}(C+P) \to k_P \to 0.$$

Taking cohomology, we have the exact sequence

$$0 \to H^0(\mathcal{L}(C)) \to H^0(\mathcal{L}(C+P)) \to H^0(k_P) \xrightarrow{\delta} H^1(\mathcal{L}(C)).$$

By definition, $H^0(k_P) = k_P(X) = k$. If $\delta = 0$, then $H^0(\mathcal{L}(C+P)) \cong H^0(k_P) \oplus H^0(\mathcal{L}(C))$. Thus $l(C+P) = l(C) + \dim H^0(k_P) = l(C) + 1$. If $H^1(\mathcal{L}(C)) \neq 0$, then the map δ is injective, since any non-zero linear map from a one-dimensional vector space is injective. This implies that $H^0(\mathcal{L}(C)) \cong H^0(\mathcal{L}(C+P))$, so $l(C) = l(C+P)$.

With the Lemma in hand, we continue with the problem at hand. Recall that maps to projective space are given by linear systems: an invertible sheaf generated by an appropriate number of sections. We can acquire invertible sheaves from divisors, where the number of generating sections is $l(D)$, and the degree of the map will be the degree of the divisor. Thus, It suffices to find a divisor D with $\deg(D) \leq g+1$ and $l(D) = 2$.

Starting with an arbitrary divisor E of degree equal to $g+1$, Riemann Roch says $l(E) - l(K-E) = 2$, so $l(E) \geq 2$. If $l(E) = 2$ we are done. If not, we can subtract a point from E to obtain a new divisor E_1. By construction, $\deg(E_1) < \deg(E)$, and by the lemma, $l(E_1)$ will either be $l(E)$ or $l(E)-1$. Note that if $\deg(D) \leq 0$, then $l(D) < 2$ by Riemann Roch. Thus, as we subtract points from our original choice of divisor E, the number of sections goes from greater than 2 to less than 2 by at most one at a time, so it must equal 2 at some point. This gives the desired divisor. □

16 Problem 36.1

Let C, D be two divisors on a surface X. Show that $C \cdot D = \chi(0) - \chi(-C) - \chi(-D) + \chi(-C - D)$.

Proof. For a divisor E on a surface X, Riemann Roch says $\chi(E) = \frac{1}{2}E \cdot (E - K) + 1 + p_a$, where p_a is the arithmetic genus of X. We apply this formula to each term in the expression $\chi(0) - \chi(-C) - \chi(-D) + \chi(-C - D)$, and note in advance that the $1 + p_a$ from each term cancels out thanks to the $+ - - +$ signs. Since $0 \cdot E = 0$ for any divisor E, we can also ignore completely the $\chi(0)$ term. Continuing, we have

$$\frac{1}{2}\left(-(-C)\cdot(-C-K) - (-D)\cdot(-D-K) + (-C-D)\cdot(-C-D-K)\right)$$
$$= \frac{1}{2}\left(-C\cdot(C+K) - D\cdot(D+K) + (C+D)\cdot(C+D+K)\right)$$
$$= \frac{1}{2}\left(-C\cdot(C+K) - D\cdot(D+K) + C\cdot(C+K) + C\cdot D + D\cdot(D+K) + D\cdot C\right)$$
$$= \frac{1}{2}\left(2C\cdot D\right) = C\cdot D.$$

□