MATH 7240 Homework 2

Andrea Bourque

October 2022

1 Problem 7.1

For chain complexes $A^\bullet, B^\bullet$ of objects in an abelian category, and a chain map $A^\bullet \to B^\bullet$, consider the triangle

$$\begin{tikzcd}
& \mathrm{Cone}^\bullet(u) \arrow[dl, "{w[1]}"'] & \\
A^\bullet \arrow[rr, "u"] & & B^\bullet \arrow[ul, "v"']
\end{tikzcd}$$

Show that $v \circ u$ and $w[1] \circ v$ are chain-homotopic to 0.

Proof. In fact, $w[1] \circ v = 0$. The map v is $v(b) = (0, b)$, and the map $w[1]$ is $w[1](a, b) = a$. Thus $w[1] \circ v(b) = w[1](0, b) = 0$.

For $v \circ u$, consider the map $f : A^\bullet \to \mathrm{Cone}^\bullet(u)[-1]$ sending a to $(a, 0)$. Then $f \circ d_A(a) = (d_A a, 0)$ and $d_{\mathrm{Cone}} \circ f(a) = \begin{pmatrix} -d_A & 0 \\ u & d_B \end{pmatrix} (a, 0) = (-d_A a, u(a))$. Thus $v \circ u(a) = (0, u(a)) = f \circ d_A(a) + d_{\mathrm{Cone}} \circ f(a)$, so $v \circ u$ is chain homotopic to 0. $\square$

2 Problem 7.2

Suppose now that $0 \to A^\bullet \xrightarrow{\alpha} B^\bullet \xrightarrow{\beta} C^\bullet \to 0$ is an exact sequence, and $\alpha = u$, the map from the previous problem. Construct a quasi-isomorphism s such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{Cone}^\bullet(u) \arrow[dl, "w[1]"'] \arrow[r, dashed, "s"] & C^\bullet \\
A^\bullet \arrow[rr, "u=\alpha"] & & B^\bullet \arrow[ul, "v"'] \arrow[u, "\beta"']
\end{tikzcd}$$

Proof. Let $s(a,b) = \beta(b)$. Since $v(b) = (0,b)$, this clearly makes the diagram commute. Then we must show s is a quasi-isomorphism. Since each β^n is surjective, each s^n is surjective, and thus so are the induced maps on cohomology. To show they are injective, consider $(a,b) \in \mathrm{Cone}^\bullet(u)$ with $d(a,b) = (-da, u(a) + db) = 0$ which maps to dc for some $c \in C^\bullet$., i.e. $\beta(b) = dc$. Since each β^n is surjective, there is some b' such that $c = \beta(b')$. Then $dc = d\beta(b') = \beta(db')$. Then $b - db' \in \ker\beta = \mathrm{im}\alpha = \mathrm{im}u$, so that there is some a' such that $b - db' = u(a')$. Applying d to both sides and using $d^2 = 0$, we have $db = du(a') = u(da')$. But $u(a) + db = 0$, so $-u(a) = u(da')$. Then $a + da' \in keru = \ker\alpha = 0$, so that $a = -da'$. Thus $(a,b) = d(a',b') = (-da', u(a') + db')$. This means that the induced maps on cohomology are injective. $\square$

3 Problem 11.3

Show that $Y \subset X = \mathrm{Spec} A$ is irreducible if and only if $I(Y) \subset A$ is a prime ideal.

Proof. Suppose Y is irreducible. Let $x, y \in A$ be such that $xy \in I(Y)$. Then $V(I(Y)+(x)) \cup V(I(Y)+(y)) = V(I(Y)^2+xI(Y)+yI(Y)+(xy)) = V(I(Y)) = Y$, where the second to last inequality uses $xy \in I(Y)$ and radical of $I(Y)^2$ is $I(Y)$, and the last inequality uses that Y is closed (irreducible subsets are always closed). Since Y is irreducible, then without loss of generality $V(I(Y)+(x)) = Y$, so that the radical of $I(Y)+(x)$ equals the radical of $I(Y)$, which is $I(Y)$. Since $x \in I(Y)+(x)$, this means $x \in I(Y)$, so $I(Y)$ is prime.

If $I(Y)$ is prime, suppose $Y = Y_1 \cup Y_2$ for proper closed subsets Y_1, Y_2 of Y. Then $I(Y)$ is properly contained in $I(Y_1)$ and $I(Y_2)$, so that there are elements $a_1 \in I(Y_1) - I(Y), a_2 \in I(Y_2) - I(Y)$. Then $a_1 a_2 \in I(Y_1) \cap I(Y_2) = I(Y_1 \cup Y_2) = I(Y)$. Since $I(Y)$ is prime, this means $a_1 \in I(Y)$ or $a_2 \in I(Y)$, a contradiction. Thus Y is irreducible. $\square$

4 Problem 12.1

Let $Y \subset \mathbb{A}^3$ be the set $\{(t, t^2, t^3) \mid t \in k\}$. Show that Y is an affine variety of dimension 1. Find generators for the ideal $I(Y)$. Show that $A(Y)$ is isomorphic to a polynomial ring in one variable over k.

Proof. We use Singular. First, we define an ideal I in $k[x, y, z, t]$ with generators $x-t, y-t^2, z-t^3$, and then use `eliminate(I,t);` to get generators without not involving t. Then, we switch to $k[x, y, z]$ and create the ideal with the generators without t, and then use `dim(std(I));` to get the dimension. It is important to have `std(I)` so that Singular can properly compute the dimension. Doing this shows that $I(Y) = (y^2 - xz, xy - z, x^2 - y)$ and the dimension is 1. In fact, from these generators we can also get the dimension by computing the Jacobian matrix and using row/column operations to show that it always has rank 2. Finally, $A(Y) = k[x, y, z]/I(Y) = k[x, x^2, x^3] = k[x]$. □

5 Problem 12.2

Let $Y \subset \mathbb{A}^3$ be defined by $x^2 - yz = 0$ and $xz - x = 0$. Show that Y is a union of three irreducible components. Describe them and find their prime ideals.

Proof. Note the second equation factors, so we get $x = 0$ or $z = 1$. If $x = 0$, then $y = 0$ or $z = 0$. If $z = 1$, then $y = x^2$. Thus, the three irreducible components are the y-axis, the z-axis, and the parabola $y = x^2$ in the $z = 1$ plane. The corresponding primes are $(x, z), (x, y), (x^2 - y, z - 1)$. This can be double-checked in Singular. First, load primdec.lib via `LIB "primdec.lib";`. Then

```
ring r = 0,(x,y,z),dp;
ideal i = x2-yz, xz-x;
list pr = primdecGTZ(i);
pr;
```

displays the primary decomposition of the ideal corresponding to Y, along with the associated primes. $\square$

6 Problem 12.3

Let $Y \subset \mathbb{A}^3$ be given parametrically by $x = t^3, y = t^4, z = t^5$. Show that $I(Y)$ is a prime ideal of height 2 in $k[x, y, z]$ which cannot be generated by 2 elements.

Proof. We again use Singular. First, we create an ideal in $k[x, y, z, t]$ generated by $x - t^3, y - t^4, z - t^5$, and then using eliminate to get rid of the t dependence. We get $I(Y) = (y^2 - xz, x^2y - z^2, x^3 - yz)$. Computing the primary decomposition of this ideal shows that it has one associated prime, $(y^5 - z^4, y^2 - xz, xy^3 - z^3, x^2y - z^2, x^3 - yz)$. At first, this may seem larger, since it has more generators. However, modulo $I(Y)$, $y^5 = y(xz)^2 = x^2yz^2 = z^2z^2 = z^4$, and $xy^3 = xy(xz) = x^2yz = z^2z = z^3$, so in fact this new ideal is just $I(Y)$. Thus $I(Y)$ is prime. Geometrically, this corresponds to Y being irreducible, which makes sense because it is just a curve parametrized by one variable. To compute its height, we must use primdecint.lib, which has the function heightZ; the height of $I(Y)$ is 2. Geometrically, the height is $\dim k^3 - \dim Y = 3 - 1$. Finally, we use mres to get a minimal resolution of $I(Y)$, which tells us the minimal number of generators; it is 3, so $I(Y)$ cannot be generated by 2 elements. $\square$

7 Problem 12.4

(a) Show that a topological space is noetherian if and only if every open set is quasi-compact.

Proof. First suppose X is a noetherian topological space. Let U be an open subset of X, and let $\{U_\alpha\}$ be an open cover of U. Consdier the ascending chain of opens given by taking unions of the U_α, i.e. each step includes a new element of the cover. An ascending chain of opens corresponds to a descending chain of their closed complements, which must terminate after finitely many steps. This means the union of the U_α does not change after some finite number of steps. This gives a finite cover of U.

Now suppose every open set of X is quasi-compact. Consider a descending chain of closed sets. This corresponds to an ascending chain of their open complements. The union of the opens in this chain is an open set with the elements of the chain forming an open cover. There is a finite sub-cover, meaning it takes only finitely many steps in the chain to get the full union. Thus the chain of closed sets terminates also. □

(b) If X is an affine scheme, show that X is quasi-compact, but not in general noetherian.

Proof. By definition, an affine scheme X is isomorphic as locally ringed space to the spectrum of a ring, say $X \cong \mathrm{Spec} A$. We have seen before that the spectrum of a ring is quasi-compact, meaning X is as well. As a proof of this fact, though, we use the basis of open sets $D(f)$ which are the complement of the closed $V(f)$. The union of sets $D(f_i)$ for $i \in I$ is the complement of the intersection of the $V(f_i)$, which is $V((f_i)|_{i\in I})$ Assuming that the $D(f_i)$ cover $\mathrm{Spec} A$, then we have that $(f_i)|_{i\in I} = (1)$, so there is a finite A-linear combination of the f_i equalling 1. Then the collection of opens corresponding to the non-zero terms in this linear combination is a finite subcover.

As an example of when $\mathrm{Spec} A$ is not noetherian, we take $A = \mathbb{C}[x_1, x_2, ...]$. Then the ascending, non-terminating chain of radical ideals $(x_1) \subset (x_1, x_2) \subset ...$ corresponds to a descending chain of closed sets, which must also be non-terminating. □

(c) If A is a noetherian ring, show that $\mathrm{Spec} A$ is noetherian.

Proof. A descending chains of closed subsets of $\mathrm{Spec} A$ correspond bijectively to ascending chains of radical ideals in A. Since A is noetherian, such ascending chains must terminate eventually, meaning the descending chains also terminate eventually. □

(d) Give an example to show that $\mathrm{Spec} A$ can be noetherian even when A is not.

Proof. Let $A = \mathbb{C}[x_1, x_2, ...]/(x_1^2, x_2^2, ...)$. Since $x_i^2 = 0 \in P$ for any prime ideal P, then $x_i \in P$, so that $(x_1, x_2, ...) = P$. Then Spec$A$ consists of a single point corresponding to the unique prime ideal $(x_1, x_2, ...)$, and so is trivially noetherian. However A itself is not noetherian since it has a non-finitely generated ideal, namely the aforementioned ideal. $\square$

8 Problem 13.1

(a) If $T_1 \subset T_2$ are subsets of S^h, then $Z(T_1) \supset Z(T_2)$.

Proof. Let $P \in Z(T_2)$. Then all elements of T_2 vanish on P. Since every element of T_1 is an element of T_2, we must have that all elements of T_1 vanish on P, so $P \in Z(T_1)$. □

(b) If $Y_1 \subset Y_2$ are subsets of $\mathbb{P}^n$, then $I(Y_1) \supset I(Y_2)$.

Proof. Let $f \in I(Y_2)$. Then f vanishes on all of Y_2. Then f must vanish on all of Y_1 as well, so $f \in I(Y_1)$. □

(c) For any two subsets Y_1, Y_2 of $\mathbb{P}^n$, we have $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

Proof. Since $Y_1, Y_2 \subset Y_1 \cup Y_2$, from part (b) we have that $I(Y_1 \cup Y_2) \subset I(Y_1)$ and $I(Y_1 \cup Y_2) \subset I(Y_2)$, so $I(Y_1 \cup Y_2) \subset I(Y_1) \cap I(Y_2)$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then let $P \in Y_1 \cup Y_2$. If $P \in Y_1$, then since $f \in I(Y_1)$, $f(P) = 0$. If $P \in Y_2$, then since $f \in I(Y_2)$, $f(P) = 0$. Thus f vanishes on $Y_1 \cup Y_2$, so $I(Y_1 \cup Y_2) \supset I(Y_1) \cap I(Y_2)$. □

(d) If $\mathfrak{a} \subset S$ is a homogeneous ideal with $Z(\mathfrak{a}) \neq \varnothing$, then $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.

Proof. By the homogeneous Nullstellensatz (Hartshorne Exercise I.2.1), $I(Z(\mathfrak{a})) \subset \sqrt{\mathfrak{a}}$. On the other hand, since $\mathfrak{a} \subset I(Z(\mathfrak{a}))$ and $I(Y)$ is always a radical ideal, we have $\sqrt{\mathfrak{a}} \subset I(Z(\mathfrak{a}))$. □

(e) For any subset $Y \subset \mathbb{P}^n$, we have $Z(I(Y)) = \overline{Y}$.

Proof. Since $Y \subset Z(I(Y))$ and $Z(I(Y))$ is closed by definition, we have $\overline{Y} \subset Z(I(Y))$. But $\overline{Y} = Z(J)$ for some ideal J, so that for any $P \in Y$ and any $f \in J$, we have $f(P) = 0$. Then $J \subset I(Y)$, so $Z(I(Y)) \subset Z(J) = \overline{Y}$. □

9 Problem 13.2

(a) There is a 1-1 inclusion reversing correspondence between algebraic sets in $\mathbb{P}^n$ and homogeneous radical ideals of S not equal to S, given by $Y \mapsto I(Y)$ and $\mathfrak{a} \mapsto Z(\mathfrak{a})$.

Proof. To show that $I(Y)$ is radical, suppose $f^n \in I(Y)$ for some integer $n > 0$. Then $f^n(P) = 0$ for all $P \in Y$. But since we are working over a field, this must mean $f(P) = 0$ for all $P \in Y$, i.e. $f \in I(Y)$. On the other hand, $Z(\mathfrak{a})$ is an algebraic set by definition. Inclusion reversing was proved in the previous problem. Finally, since $I(Z(\mathfrak{a})) = \mathfrak{a}$ for radical ideals and $Z(I(Y)) = Y$ for algebraic sets, the correspondence is 1-1. □

(b) An algebraic set $Y \subset \mathbb{P}^n$ is irreducible if and only if $I(Y)$ is a prime ideal.

Proof. The proof is the exact same as in Problem 11.3. □

(c) Show that $\mathbb{P}^n$ itself is irreducible.

Proof. $I(\mathbb{P}^n) = (0)$, which is clearly prime. □

10 Problem 13.3

If $Y \subset \mathbb{A}^n$ is an affine variety, we identify $\mathbb{A}^n$ with an open set $U_0 \subseteq \mathbb{P}^n$ by the homeomorphism φ_0. Then we can speak of the closure of Y in $\mathbb{P}^n$, called the projective closure.

(a) Show that $I(\overline{Y})$ is the ideal generated by $\beta(I(Y))$, using the notation of the proof of (2.2).

Proof. Here $\beta : k[x_1, ..., x_n] \to k[X_0, ..., X_n]^h$ sends polynomial g of degree e to the homogeneous polynomial $X_0^e g(X_1/X_0, ..., X_n/X_0)$. Note that $\overline{Y} = Z(I(\varphi_0^{-1}(Y)))$, so that $I(\overline{Y}) = I(\varphi_0^{-1}(Y))$, the set of homogeneous polynomials which vanish on the image of Y in $\mathbb{P}^n$. This clearly contains $\beta(I(Y))$. If $F \in I(\overline{Y})$, then F can be de-homogenized on U_0 to give $f \in I(Y)$. It may be the case that $\beta(f) \neq F$. However, $Z(F) \cap U_0 = \varphi_0^{-1}(V(f)) = Z(\beta(f)) \cap U_0$, and $Z(\beta(f))$ is the projective closure of $V(f)$, so $Z(\beta(f)) \subset Z(F)$, and so $(F) \subset (\beta(f))$, i.e. F is generated by $\beta(f)$. Thus $I(\overline{Y})$ is generated by $\beta(I(Y))$. □

(b) Let $Y \subseteq \mathbb{A}^3$ be the twisted cubic of Exercise I.1.2 (Problem 12.1). Find generators for $I(Y)$ and $I(\overline{Y})$, and use this example to show that if $f_1, ..., f_r$ are generators of $I(Y)$, then $\beta(f_1), ..., \beta(f_r)$ are not necessarily generators of $I(\overline{Y})$.

Proof. Write $I(Y) = (x^3 - z, x^2 - y)$; we previously wrote it with 3 generators, but it can also be written with just these two. The curve Y is mapped to the set of points $[1 : t : t^2 : t^3]$. Projectivizing gives the set of points $[s^3 : s^2t : st^2 : t^3]$, so that $\overline{Y}$ is just Y with an extra point $[0 : 0 : 0 : 1]$. We can find $I(\overline{Y})$ in singular by creating the ideal $(w - s^3, x - s^2t, y - st^2, z - t^3)$ and then eliminating s, t. This gives $I(\overline{Y}) = (x_2^2 - x_1x_3, x_1x_2 - x_0x_3, x_1^2 - x_0x_2)$. Using mres, we can see that the minimal number of generators for $I(\overline{Y})$ is 3, so that it cannot be generated by $\beta(x^3 - z), \beta(x^2 - y)$. □