MATH 7240 Homework 1

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1 Problem 1.2

Let $\mathcal{F}$ be a sheaf of abelian groups on X, and let $s \in \mathcal{F}(U)$ for an open subset U. The support of s is defined to be $\{P \in U \mid s_P \neq 0\}$. Show that $\mathrm{Supp}(s)$ is a closed subset of U.

Proof. It is equivalent to show that $S = \{P \in U \mid s_P = 0\}$ is open. If S is empty, it is open. Otherwise, pick $P \in S$. $s_P = 0$ means that there is an open neighborhood W of P such that $s|_W = 0$, and therefore $s_Q = 0$ for all $Q \in W$. Thus S is open. $\square$

2 Problem 1.3

Let $\mathcal{F}, \mathcal{G}$ be sheaves of abelian groups on X. For any open U, show that the set $\mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ has a natural structure of an abelian group. Show that the presheaf $U \mapsto \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is a sheaf.

Proof. Addition of maps is defined "setwise", i.e. for $\eta, \nu : \mathcal{F}|_U \to \mathcal{G}|_U$, $\eta + \nu$ is the morphism such that $(\eta + \nu)(V) = \eta(V) + \nu(V) : \mathcal{F}|_U(V) \to \mathcal{G}|_U(V)$ for all opens $V \subseteq U$. Thus the zero morphism $0 : \mathcal{F}|_U \to \mathcal{G}|_U$ is the morphism such that $0(V) : \mathcal{F}|_U(V) \to \mathcal{G}|_U(V)$ is the trivial homomorphism for all opens $V \subseteq U$.

Now consider an open covering $\{U_i\}_{i \in I}$ of U and elements $\eta_i \in \mathrm{Hom}(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i})$ which agree on intersections. We want natural maps $\eta(V) : \mathcal{F}(V) \to \mathcal{G}(V)$ for all opens $V \subseteq U$. We define $\eta(V)(s)$ to be the section obtained from gluing together the sections $\eta_i(V \cap U_i)(s|_{V \cap U_i})$. These sections can be glued together because the maps η_i agree on intersections and because $\mathcal{F}, \mathcal{G}$ are sheaves. $\square$

3 Problem 5.2

Using Čech cohomology, compute $H^i(X, \mathbb{Z})$ for compact Riemann surface X.

Proof. First we do the sphere. We choose as a covering 5 open sets which are described as follows: two caps which are diametrically opposed but do not pass the equator (and hence do not overlap), and three rectangular strips around the equator. Calling these U_1 through U_5, (with U_1, U_5 the caps), the nonempty intersections of two opens are $U_{12}, U_{13}, U_{14}, U_{23}, U_{24}, U_{34}, U_{25}, U_{35}, U_{45}$. The nonempty intersections of three opens are $U_{123}, U_{124}, U_{134}, U_{235}, U_{245}, U_{345}$. The intersection of any four opens is empty. Notice that all finite intersections are contractible; this is a "good cover." We use without proof that for a good cover, the Čech cohomology for the cover agrees with the Čech cohomology of the space. Now, $C^0 = \mathbb{Z}^5, C^1 = \mathbb{Z}^9, C^2 = \mathbb{Z}^6$, and further complexes are trivial. The differential $d : C^0 \to C^1$ acts as

$$d(s, t, x, y, z) = (t - s, x - s, y - s, x - t, y - t, z - t, ...).$$

It is evident that the kernel of this map is the diagonal $\{(x, x, x, x, x) \mid x \in \mathbb{Z}\} \cong \mathbb{Z}$. The next differential acts on $C^1 = \mathbb{Z}^9$ via the following matrix:

$$\begin{bmatrix} 1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{bmatrix}.$$

Putting this matrix in Smith normal form shows that the kernel of the differential is $\mathbb{Z}^9/\mathbb{Z}^5 = \mathbb{Z}^4$. Thus $H^0 = \mathbb{Z}/0 = \mathbb{Z}$, $H^1 = \mathbb{Z}^4/\mathbb{Z}^{5-1} = 0$, $H^2 = \mathbb{Z}^6/\mathbb{Z}^{9-4} = \mathbb{Z}$. This agrees with, for instance, the singular (co)homology of the sphere.

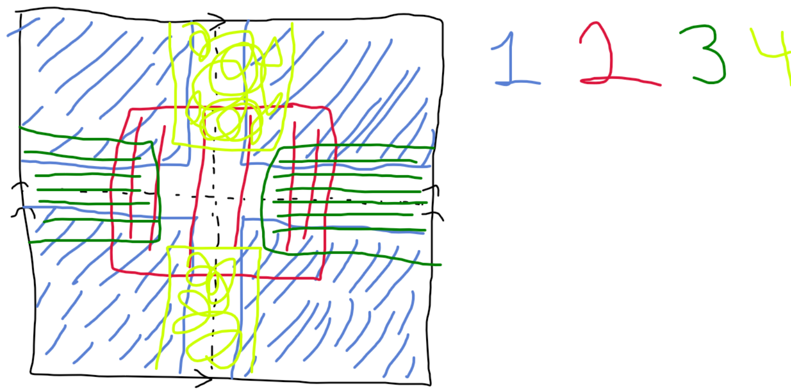

Figure 1: Cover for torus

Now we do the torus. Consider the beautiful figure 1, which illustrates the cover. As indicated, there are four opens. This time, finite intersections are not contractible, but are finite unions of contractible sets, which is still sufficient for the cohomology of the cover to agree with the cohomology of the space. As there are four connected opens, $C^0 = \mathbb{Z}^4$. For the intersections of two at a time, we have $12 = \mathbb{Z}^4, 13 = \mathbb{Z}^2, 14 = \mathbb{Z}^2, 23 = \mathbb{Z}^2, 24 = \mathbb{Z}^2, 34 = \varnothing$, so $C^1 = \mathbb{Z}^{12}$. Next, $123 = \mathbb{Z}^4, 124 = \mathbb{Z}^4, 134 = \varnothing, 234 = \varnothing$, so $C^2 = \mathbb{Z}^8$. Finally, the intersection of all four opens together is empty. Now we write down the differentials. The first differential is

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}.$$

The kernel is clearly $\mathbb{Z}$. The next differential is

$$\begin{bmatrix}
1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.$$

Putting this in Smith normal form shows that the kernel is $\mathbb{Z}^5$. Thus, $H^0 = \mathbb{Z}/0 = \mathbb{Z}, H^1 = \mathbb{Z}^5/\mathbb{Z}^{4-1} = \mathbb{Z}^2, H^2 = \mathbb{Z}^8/\mathbb{Z}^{12-5} = \mathbb{Z}$.

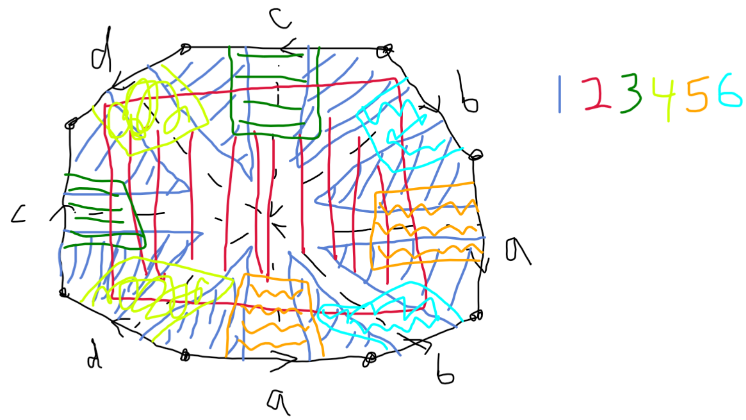

Figure 2: Genus 2 covering

Using the previous work and the pictures of genus 1 and 2 coverings, we now show the result for arbitrary positive genus. The covering we consider is as follows (all opens described are contractible): U_1 is a neighborhood of the point which represents the vertices of the $4g$-gon; U_2 sits in the middle of the $4g$-gon; U_j for $j = 3, ..., 2g+2$ are strips which sit on the $2g$ different identified edges, intersecting both U_1 and U_2 in two disjoint contractible strips; furthermore, the U_j are pairwise disjoint. The last condition implies that the only non-zero elements of the Čech complex are C^0, C^1, C^2.

First, C^0 is just $\mathbb{Z}^{2g+2}$, with one copy of $\mathbb{Z}$ coming from each open in the cover. Next, $C^1 = \mathbb{Z}^{12g}$, with a copy of $\mathbb{Z}^{4g}$ coming from $U_1 \cap U_2$, and $4g$ copies of $\mathbb{Z}^2$ coming from $U_1 \cap U_j, U_2 \cap U_j$ for $j > 2$. Finally, $C^2 = \mathbb{Z}^{8g}$, with $2g$ copies of $\mathbb{Z}^4$ coming from $U_1 \cap U_2 \cap U_j$ for $j > 2$.

The differential $d : C^0 \to C^1$ has a simple description. For $x \in C^0$, the first $4g$ components of dx are $x_2 - x_1$. The next $4g$ components come in $2g$ pairs $(x_j - x_1, x_j - x_1)$ for $j > 2$. The last $4g$ components also come in $2g$ pairs $(x_j - x_2, x_j - x_2)$ for $j > 2$. It is then clear that the kernel is $\mathbb{Z}$, as $x_2 - x_1 = x_j - x_1 = 0$ for $j > 2$ implies $x_1 = x_2 = x_j$, so that x is determined by a single integer. Then $H^0 = \mathbb{Z}$.

The differential $d : C^1 \to C^2$ is significantly more complicated. More specifically, the issue comes from the $12 \to 12j$ restriction map corresponding to $U_1 \cap U_2 \cap U_j \subset U_1 \cap U_2$. The $1j \to 12j$ and $2j \to 12j$ restriction maps are simpler: they are $(x, y) \mapsto (x, y, y, x)$ and $(x, y) \mapsto (x, x, y, y)$ respectively. The $12 \to 12j$ restriction map sends $x \in \mathbb{Z}^{4g}$ to $(x_k, x_{k+1}, x_{k+2}, x_{k+3})$, where $k = 2j - \frac{11+(-1)^j}{2}$ and the index is taken mod $4g$ if necessary (this only occurs for $k + 3 = 4g + 1$ when $j = 2g+2$). Let $w = (x, y^3, ..., y^{2g+2}, z^3, ..., z^{2g+2}) \in \mathbb{Z}^{12g}$, where $x \in \mathbb{Z}^{4g}$

and each y^j, z^j is in $\mathbb{Z}^2$. Then

$$(dw)_{12j} = (x_k - y_1^j + z_1^j, x_{k+1} - y_2^j + z_1^j, x_{k+2} - y_2^j + z_2^j, x_{k+3} - y_1^j + z_2^j),$$

where again $k = 2j - \frac{11+(-1)^j}{2}$ and the index of x is taken mod $4g$ if necessary. The last hurdle is to show that the kernel of this differential is $\mathbb{Z}^{4g+1}$ - and I am not sure of any way to do this. Assuming this is true, we get $H^1 = \mathbb{Z}^{4g+1}/\mathbb{Z}^{2g+2-1} = \mathbb{Z}^{2g}$, and $H^2 = \mathbb{Z}^{8g}/\mathbb{Z}^{12g-(4g+1)} = \mathbb{Z}$. $\square$

4 Problem 6.1

(a) Show that a constant sheaf on an irreducible topological space is flasque.

Proof. Let X be irreducible. Let U be a nontrivial proper open subset of X. If $U = U_1 \cup U_2$ for $U_i = U \cap X_i$ with X_i closed subsets of X, then $X = X_1 \cup X_2 \cup (X \setminus U)$. By hypotheses on U and X, we have $X_1 \cup X_2 = X$. Again, X is irreducible, so one of X_1, X_2 is in fact equal to X. This shows that one of U_1, U_2 is U, so U is irreducible. Note that irreducible spaces are also connected. It follows that a constant sheaf on X can only take two values: 0 and A, corresponding to the empty set and non-empty subsets respectively. Thus any restriction map is either $\mathrm{id} : A \to A$ or $0 : A \to 0$, both of which are surjective. □

(b) If $0 \to \mathcal{F}' \xrightarrow{\alpha} \mathcal{F} \xrightarrow{\beta} \mathcal{F}'' \to 0$ is an exact sequence of sheaves and $\mathcal{F}'$ is flasque, then for any open set U, $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U) \to 0$ is also exact.

Proof. It is always true, regardless of the condition on $\mathcal{F}'$, that $0 \to \mathcal{F}'(U) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is exact. It suffices to show that $\mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective. Let $s \in \mathcal{F}''(U)$. We know that for all $x \in U$, the map $\mathcal{F}_x \to \mathcal{F}''_x$ is surjective. Thus, we can pull $[(U, s)] \in \mathcal{F}''_x$ back to some $[(U_x, s_x)] \in \mathcal{F}_x$. Without loss of generality, we can assume that $U_x \subset U$, so that s_x maps to s restricted to U_x. Note that for $x \neq y$, we have $\beta(U_x \cap U_y)(s_x|_{xy} - s_y|_{xy}) = 0$, so that there is a $t_{xy} \in \mathcal{F}'(U_x \cap U_y)$ which maps to $s_x|_{xy} - s_y|_{xy}$. $\mathcal{F}'$ being flasque means we can pull t_{xy} up to $t'_{xy} \in \mathcal{F}'(U_y)$. Then $\alpha(U_y)(t'_{xy}) + s_y \in \mathcal{F}(U_y)$ agrees with $s_x \in \mathcal{F}(U_x)$ on the intersection, and they can be glued together to $s_{xy} \in \mathcal{F}(U_x \cup U_y)$. In particular, s_{xy} maps to s restricted to $U_x \cup U_y$.

What we have seen thus far is that for a point $x \in X$, we have an open neighborhood $U_x \subset U$ and a section $s_x \in \mathcal{F}(U_X)$ which maps to the section $s \in \mathcal{F}''(U)$ restricted to U_x, and furthermore, we can enlarge these "local preimages". The "local preimages" form a poset under inclusion, and chains have upper bounds by taking unions. By Zorn's lemma, there is a maximal element. This must be (U, s') for s' a preimage of s, because if (V, t) was maximal and $V \neq U$, we could take a point $x \in U - V$ and enlarge V to $V \cup U_x$ for some neighborhood of x. Thus, we can find a preimage of s. □

(c) If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of sheaves and $\mathcal{F}, \mathcal{F}'$ are flasque, then $\mathcal{F}''$ is flasque.

Proof. Let $V \subset U$ be opens. Let $s \in \mathcal{F}''(V)$. We have just shown that $\mathcal{F}(V) \to \mathcal{F}''(V)$ is surjective, so we can pull s back to some $t \in \mathcal{F}(V)$. Since $\mathcal{F}$ is flasque, we can pull t back to $t' \in \mathcal{F}(U)$. Then we map t' to $s' \in \mathcal{F}''(U)$. By definition of morphism of (pre)sheaves, they are compatible with restrictions, so that s' restricts to s. Thus $\mathcal{F}''(U) \to \mathcal{F}''(V)$ is surjective, so $\mathcal{F}''$ is flasque. □

(d) If $f : X \to Y$ is continuous and $\mathcal{F}$ is a flasque sheaf on X, then $f_*\mathcal{F}$ is flasque.

Proof. If $V \subset U \subset Y$, then $f^{-1}V \subset f^{-1}U \subset X$. Then the restriction map $\mathcal{F}(f^{-1}U) \to \mathcal{F}(f^{-1}V)$ is surjective, which is the restriction map $f_*\mathcal{F}(U) \to f_*\mathcal{F}(V)$. □

(e) Let $\mathcal{F}$ be a sheaf on X. Consider $\mathcal{G}$ such that $\mathcal{G}(U)$ is the set of maps $s : U \to \bigsqcup_{P \in U} \mathcal{F}_P$ such that $s(P) \in \mathcal{F}_P$ for all $P \in U$. Show that $\mathcal{G}$ is a flasque sheaf, and that there is a natural injective morphism $\mathcal{F} \to \mathcal{G}$.

Proof. Let $V \subset U$ be opens. Given $s \in \mathcal{G}(V)$, we can extend it to $s' \in \mathcal{G}(U)$ by $s'(P) = s(P)$ for $P \in V$, $s'(P) = 0 \in \mathcal{F}_P$ for $P \notin V$. Thus $\mathcal{G}$ is flasque.

Now define $\mathcal{F} \to \mathcal{G}$ by sending $t \in \mathcal{F}(U)$ to s which maps $P \in U$ to $[(U,t)] \in \mathcal{F}_P$. If $s = 0$, this means that for all $P \in U$, there is a U_P with $P \in U_P \subset U$ such that t restricts to 0 on U_P. This collection $\{U_P\}_{P \in U}$ is a cover of U. By the gluing axiom, $t = 0$, so the map $\mathcal{F} \to \mathcal{G}$ is injective. □

5 Problem 6.2

Let X be a topological space, let P be a point, and let A be an abelian group. Define a sheaf $i_P(A)$ on X as follows: $i_P(A)(U) = A$ if $P \in U$, 0 otherwise. Verify that the stalk of $i_P(A)$ is A at every point Q in the closure of P, and 0 elsewhere. Show that this sheaf can also be described as $i_*(A)$ for A the constant sheaf on the closure of P, and i is the inclusion map.

Proof. Let S be the closure of P. If U is an open set not containing P, then $X - U$ is a closed set containing P. Then $S \subseteq X - U$, or $S \cap U = \varnothing$. Then for any $Q \in S$, any open neighborhood of Q contains P. Thus the stalk of $i_P(A)$ at Q is the direct limit of copies of A, and is thus just A.

As a slight rephrasing of the previous argument, if an open set U intersects S at all, it must contain P. In particular, if $P \notin U$, then $i_*(A)(U) = 0$, since in that case $i^{-1}(U) = S \cap U = \varnothing$. If $P \in U$, then $i^{-1}(U) = S \cap U$ must be connected; if it were the disjoint union of two closed sets, one of them must contain P, and hence must contain S. Then $i_*(A)(U) = A$, since the value of the constant sheaf A on a connected set is A. $\square$