MATH 7230 Homework 5

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1 On Hecke Operators

Let M_k be the space of modular forms of weight k on $PSL_2(\mathbb{Z})$ and T_n, $n \geq 1$, the Hecke operators on M_k. Fix a subspace $V \leq M_k$, which is stable under the action of all T_n. Let

$$\mathcal{H} = \mathcal{H}_{\mathbb{C}}(V) := \mathbb{C}[T_n|_V : n \geq 1]$$

be the $\mathbb{C}$-algebra generated by the endomorphisms T_n acting on V. Define the pairing

$$\mathcal{H} \times V \to \mathbb{C}, \qquad (T, f) = a_1(T(f)),$$

where $a_1(g)$ is the coefficient of q in the q-expansion of g.

Problem 1

Show that the pairing gives an isomorphism $\mathcal{H} \cong \mathrm{Hom}(V, \mathbb{C})$. Hence $\dim \mathcal{H} = \dim V$.

Proof. We have a map $\varphi : \mathcal{H} \to \mathrm{Hom}(V, \mathbb{C})$ given by $\varphi(T)(f) = a_1(T(f))$. Since T and a_1 are linear, so is φ. Now, note that by the relations $T_m T_n = T_{mn}$ for $\gcd(m, n) = 1$ and $T_{p^{r-1}} T_p = T_{p^r} + p^{k-1} T_{p^{r-2}}$ for prime p, any element of $\mathcal{H}$ can be written as a linear combination of Hecke operators, not just compositions of Hecke operators. Now, suppose $T = c_1 T_1 + \cdots + c_n T_n$ is in $\ker \varphi$. That is, for all $f \in V$, we have $a_1(T(f)) = 0$. Since V is a finite dimensional subspace of M_k, which is a Hilbert space with respect to the Petersson inner product, V is also a Hilbert space. By the spectral theorem, V has a basis $f_1, \ldots, f_d$ of (normalized) eigenforms. Now, we have

$$\begin{aligned} 0 = a_1(T(f_i)) &= a_1(c_1 T_1(f_i) + \cdots + c_n T_n(f_i)) \\ &= a_1(c_1 a_1(f_i) f_i + \cdots + c_n a_n(f_i) f_i) \\ &= (c_1 a_1(f_i) + \cdots + c_n a_n(f_i)) a_1(f_i). \end{aligned}$$

Since f_i is a normalized eigenform, $a_1(f_i) = 1 \neq 0$, so we have $c_1 a_1(f_i) + \cdots + c_n a_n(f_i) = 0$ for each i. Thus $T(f_i) = (c_1 a_1(f_i) + \cdots + c_n a_n(f_i)) f_i = 0$. Since

T is zero on each basis vector, T is zero. Thus, φ is injective.

It follows that $\mathcal{H}$ is finite dimensional, and in particular, $\dim \mathcal{H} \leq \dim V$. If $\dim \mathcal{H} = \dim V$, then φ is an isomorphism. To prove this, we show that $\psi : V \to \mathrm{Hom}(\mathcal{H}, \mathbb{C})$ given by $\psi(f)(T) = a_1(T(f))$ is injective. It is linear, since a_1 and T are linear. Now, suppose $f \in \ker \psi$, i.e. $a_1(T(f)) = 0$ for all $T \in \mathcal{H}$. In particular, $a_1(T_m(f)) = 0$ for all $m \geq 1$. But $a_1(T_m(f)) = a_m(f) = 0$, so f is constant. Unless $k = 0$, the only constant modular form is 0, so we are done in that case. In the case $k = 0$, $M_k = \mathbb{C}$ is one-dimensional, so either $V = 0$ or $V = \mathbb{C}$. If $V = 0$, then since $0 \leq \dim \mathcal{H} \leq \dim V = 0$, we must have $\dim \mathcal{H} = 0$, so φ is an isomorphism. If $V = \mathbb{C}$, then we can use the fact that $\mathcal{H}$ has the non-zero (identity) element T_1 to say $1 \leq \dim \mathcal{H}$. Since we already have $\dim \mathcal{H} \leq \dim V = 1$, we have $\dim \mathcal{H} = \dim V$, so φ is an isomorphism. $\square$

Problem 2

Let $V = S_k$, the subspace of cusp forms. Show that the first $d := \dim S_k$ Hecke operators span the space $\mathcal{H}$.

Proof. I'm not sure if we covered this in class, but note that by induction on k (using $\Delta M_{k-12} = S_k$ and $M_k = \mathbb{C}E_k \oplus S_k$), there is a basis $f_1, \ldots, f_d$ for S_k for which $a_i(f_i) = 1$ and $a_i(f_j) = 0$ for $i < j$. In particular, this implies that if $f \in S_k$ satisfies $a_i(f) = 0$ for $i = 1, \ldots, d$, then $f = 0$.

Now, let $f_1, \ldots, f_d$ be a basis of normalized eigenforms. Since $\dim \mathcal{H} = d$, the operators $T_1, \ldots, T_d$ either span $\mathcal{H}$ or they are linearly dependent. Suppose the latter; let $c_1 T_1 + \cdots + c_d T_d = 0$ on S_k, with some $c_j \neq 0$. Applying both sides of the equation to the basis elements, we get $c_1 a_1(f_i) + \cdots + c_d a_d(f_i) = 0$ for each i. This means that the matrix A with (i, j) component equal to $a_i(f_j)$ satisfies $cA = 0$, where c is the row vector with entries c_i. Thus A is not full rank, so there is some non-zero column vector c' such that $Ac' = 0$. Note that the ith component of Ac' is $c'_1 a_i(f_1) + \cdots + c'_d a_i(f_d) = a_i(c'_1 f_1 + \cdots + c'_d f_d)$. In particular, the cusp form $f = c'_1 f_1 + \cdots + c'_d f_d$ satisfies $a_i(f) = 0$ for each $i = 1, \ldots, d$. By the previous observation, this means $f = 0$. Since the f_i are a basis, this means $c' = 0$, a contradiction. $\square$

2 On Congruence Subgroups

Problem 1

Draw/Find a fundamental domain for the groups $\Gamma(2)$, $\Gamma_1(3)$, and $\Gamma_0(4)$. In the fundamental domain that you choose, locate the vertices, cusps, and elliptic points.

Proof. Recall that we can give a fundamental domain by taking the translates of $PSL_2(\mathbb{Z})$ fundamental domain by right coset representatives of the subgroup in $PSL_2(\mathbb{Z})$. We use Sagemath to find right coset representatives for the various groups (see Problem 3 for the command). For $\Gamma(2)$ we have

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Writing this in terms of the generators S and T, we have I, T, S, ST, TS, TST. Therefore, the fundamental domain is given as in Figure 1. The cusps can be seen as the vertices which touch $\mathbb{Q} \cup \{\infty\}$. The cusps $0, 1$ in $\mathbb{Q}$ have been labeled; there is also a cusp at ∞. They are inequivalent, which can be argued directly as follows. The result of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acting on 0 is b/d. In the case of $\Gamma(2)$, b is even and d is odd, so b/d can never be 1 or ∞. Similarly, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \infty = a/c$ is an odd divided by an even, which can never be 0 or 1. The vertices are the other labeled points, including the cusps. There are no elliptic points, which can be argued directly as follows. We know that elliptic matrices have trace bounded in magnitude by the number 2. The diagonal entries of elements in $\Gamma(2)$ are both odd, by definition, so the only possible elliptic matrices in $\Gamma(2)$ must have trace 0. However, analyzing $\det \begin{pmatrix} 2a+1 & 2b \\ 2c & -2a-1 \end{pmatrix} = 1 \bmod 4$ gives $-1 \equiv 1 \bmod 4$, a contradiction. Thus, $\Gamma(2)$ has no elliptic elements, hence no elliptic points.

For $\Gamma_1(3)$, the right coset representatives are I, S, TST, ST^2. I couldn't find or make a good picture that included the ST^2 translation of the standard fundamental domain, so I will describe it explicitly. It is also displayed in Figure 2. We have $ST^2 = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix}$, to the region given by ST^2 will be the hyperbolic triangle with vertices

$$\frac{-1}{i\infty + 2}, \frac{-1}{\frac{1}{2} + i\frac{\sqrt{3}}{2} + 2}, \frac{-1}{-\frac{1}{2} + i\frac{\sqrt{3}}{2} + 2},$$

which simplifies to

$$0, \frac{-5}{14} + i\frac{\sqrt{3}}{14}, \frac{-1}{2} + i\frac{\sqrt{3}}{6}.$$

We note that 0 and $\frac{-1}{2} + i\frac{\sqrt{3}}{6}$ are also vertices of the ST region, so the ST^2 region shares an edge with the ST region. Visually, we see that there are cusps at $0, 1, i\infty$. However, 0 and 1 are equivalent, since $T \in \Gamma_1(3)$ and $T0 = 1$. On the other hand, the image of ∞ is a non-multiple of 3 divided by a multiple of 3, so it can't be 0 or 1. Thus, we have 2 non-equivalent cusps, 0 and ∞. Finally, we use the MAGMA command `EllipticPoints(CongruenceSubgroup(1,3));` to find that there is an elliptic point at $\frac{1}{2} + i\frac{\sqrt{3}}{6}$. This is a T translate of the point $\frac{-1}{2} + i\frac{\sqrt{3}}{6}$, which we earlier identified as a common vertex of the ST and ST^2 regions. Both of these points have been marked separately, despite being equivalent. The order of this elliptic point is 3, since the Sagemath command `Gamma1(3).nu2()` shows that there are no elliptic points of order 2.

For $\Gamma_0(4)$, the right coset representatives are $I, S, TST, ST^2, ST^3, ST^{-2}S^3$. I will make no attempt to illustrate or list out the vertices for this one, but I will point out that all of the regions except TST are have absolute value of the real part bounded by 1/2. We can argue directly to show that there are no elliptic points as follows. Analyzing $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \bmod 4$, for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$, shows that $ad \equiv 1 \bmod 4$. This is only possible if $a \equiv d \bmod 4$ and $a \equiv d \equiv 1 \bmod 2$. In particular, $a + d \equiv 2 \bmod 4$, so $|a + d|$ cannot be less than 2. Thus, $\Gamma_0(4)$ has no elliptic elements, hence no elliptic points. We fully rely on Sagemath for the cusps. In particular, `list(Gamma0(4).cusps())` returns `[0, 1/2, Infinity]`, meaning that there are three non-equivalent cusps at $0, 1/2, \infty$.

$\square$

Figure 1: Fundamental domain for $\Gamma(2)$.

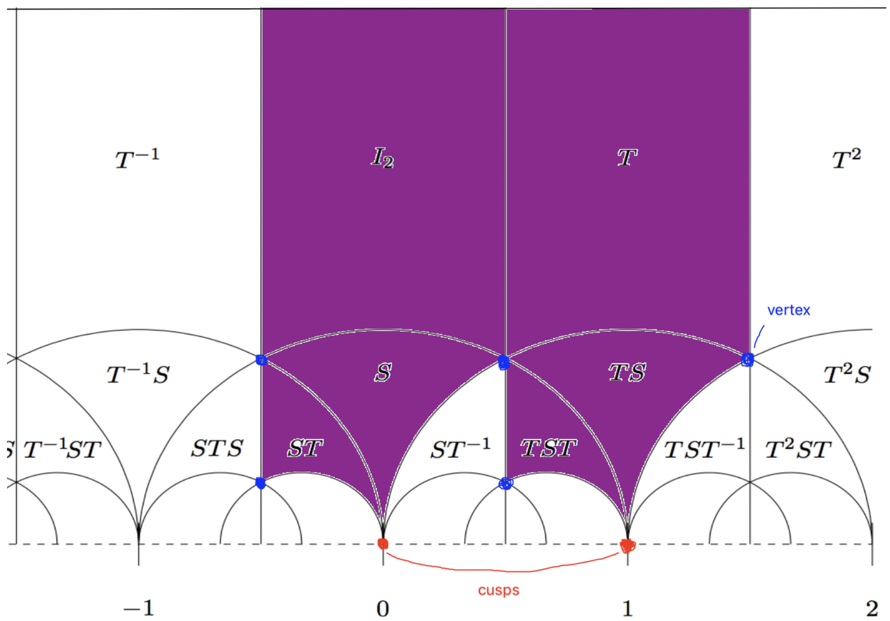
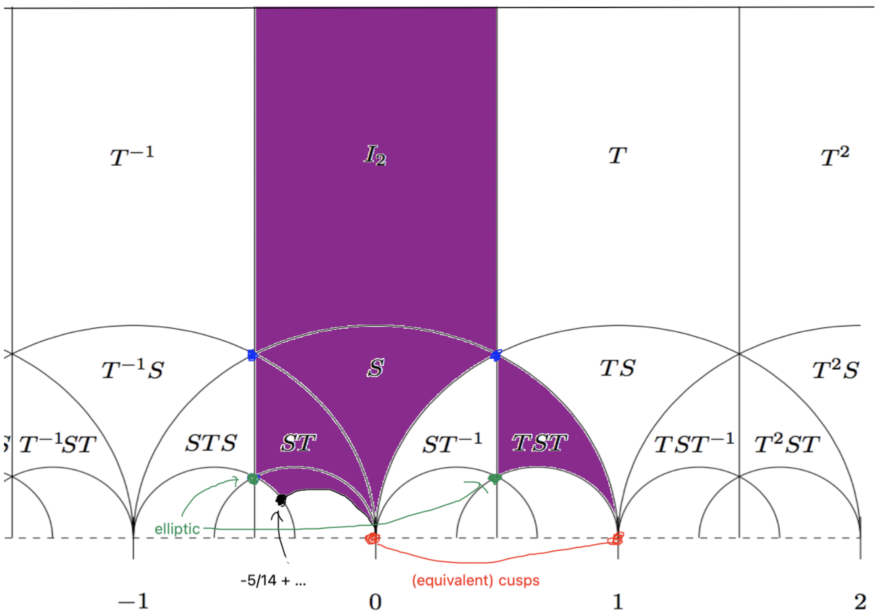


Figure 2: Fundamental domain for $\Gamma_1(3)$.



Problem 2

Is the congruence subgroup $\Gamma(2)$ conjugate to $\Gamma_0(4)$ in $SL_2(\mathbb{R})$? Justify your answer.

Proof. Yes, they are conjugate. Let $g = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}$. We claim that $\Gamma(2) = g\Gamma_0(4)g^{-1}$. It suffices to show that the conjugates of generators of $\Gamma_0(4)$ are generators of $\Gamma(2)$. Using the Sagemath command `Gamma0(4).gens()`, we see that $\Gamma_0(4)$ is generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$, and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Similarly, $\Gamma(2)$ is generated by $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$, and $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. Since $-I$ is a scalar matrix, it is conjugation invariant. We have

$$\begin{aligned}
\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} &= \begin{pmatrix} \sqrt{2} & \sqrt{2} \\ 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \\
\begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} &= \begin{pmatrix} 3\sqrt{2} & -\sqrt{2} \\ 4/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \\
&= \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}.
\end{aligned}$$

Thus $\Gamma(2) = g\Gamma_0(4)g^{-1}$ as desired. □

Problem 3

Find a set of right coset representatives of the following subgroups in $PSL_2(\mathbb{Z})$: $\Gamma_0(6)$ and $\Gamma_1(6)$.

Proof. The Sagemath commands `list(Gamma0(6).coset_reps())` and `list(Gamma1(6).coset_reps())` compute right coset representatives in $SL_2(\mathbb{Z})$. To get right coset representatives in $PSL_2(\mathbb{Z})$, we just ignore the representatives that are the negative of another representative. For $\Gamma_0(6)$ we also have

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 3\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 4\end{pmatrix},$$
$$\begin{pmatrix}0 & -1\\ 1 & 5\end{pmatrix}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}, \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}, \begin{pmatrix}-1 & -1\\ 3 & 2\end{pmatrix}.$$

For $\Gamma_1(6)$ we have

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}, \begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 2\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 3\end{pmatrix}, \begin{pmatrix}0 & -1\\ 1 & 4\end{pmatrix},$$
$$\begin{pmatrix}0 & -1\\ 1 & 5\end{pmatrix}, \begin{pmatrix}1 & 0\\ 2 & 1\end{pmatrix}, \begin{pmatrix}1 & 1\\ 2 & 3\end{pmatrix}, \begin{pmatrix}1 & 2\\ 2 & 5\end{pmatrix}, \begin{pmatrix}1 & 0\\ 3 & 1\end{pmatrix}, \begin{pmatrix}-1 & -1\\ 3 & 2\end{pmatrix}.$$

Note that it makes sense for them to have the same number of distinct cosets, because $[\Gamma_0(6) : \Gamma_1(6)] = 6(1-1/2)(1-1/3) = 2$, and $\Gamma_0(6)$ contains $-I$ while $\Gamma_1(6)$ doesn't. $\square$

Problem 4

Let p be a prime. Find a set of right coset representatives of $\Gamma_0(p)$ in $PSL_2(\mathbb{Z})$.

Proof. By the index formula, and since $-I \in \Gamma_0(p)$, we know that there will be $p(1 + 1/p) = p + 1$ distinct cosets. We claim that there are distinct coset representatives given by $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 0 & -1 \\ 1 & p-1 \end{pmatrix}$. None of the last p are equivalent to the identity, since they are not in $\Gamma_0(p)$, since $1 \not\equiv 0 \mod p$. Now, using the proposition on slide 12 of the modular group class slides, two matrices are in the same right coset of $\Gamma_0(p)$ iff $c_1 d_2 \equiv c_2 d_1 \mod p$, where c_i, d_i represent the second row of the two matrices. In the last p matrices, we have $c_1 = c_2 = 1$, so two matrices from the last p matrices in our list are in the same right coset iff their bottom right entries are equivalent mod p. Since $0, 1, \dots, p-1$ are not equivalent mod p, these matrices are not in the same right coset. Thus we have $p+1$ matrices in different cosets, so we have all of them. □

Problem 5

Let G be a finite-index subgroup of $\Gamma = SL_2(\mathbb{Z})$. For a given cusp s, let G_s be the isotropy subgroup of G at s. Show that

(a) $\mathrm{width}(s) = [\overline{\Gamma}_s : \overline{G}_s]$, and it does not depend on the choice of the representative of the cusp or the choice of element $\sigma \in SL_2(\mathbb{Z})$ such that $\sigma\infty = s$.

(b) If $\{\gamma_1, \ldots, \gamma_m\}$ is a system of representatives of right cosets $\overline{G} \setminus \overline{\Gamma}$, then $[\overline{\Gamma}_\infty : \overline{G}_\infty]$ equals the number of j such that $\gamma_j\infty = g\infty$ for some $g \in G$.

(c) If $\{s_1, \ldots, s_{\nu_\infty}\}$ is a system of representatives of the cusps of G and if $\{w_j\}$ denotes their cusp widths, then $\sum_{j=1}^{\nu_\infty} w_j = [\overline{\Gamma} : \overline{G}]$. Moreover, if $\overline{G} \lhd PSL_2(\mathbb{Z})$, then all the widths are equal.

(d) If $G_1 \leq G_2$ are subgroups of $SL_2(\mathbb{Z})$, then $\mathrm{width}_{G_2}(s) \mid \mathrm{width}_{G_1}(s)$.

Proof. (a) We first show that $\mathrm{width}(s)$ is independent of σ. First suppose $s = \infty$. $\sigma\infty = \infty$ means that σ is upper triangular. The only upper triangular matrices in $SL_2(\mathbb{Z})$ are $\pm T^n$. Thus $\pm\sigma T^m \sigma^{-1} = \pm T^{n+m-n} = \pm T^m$, so $\mathrm{width}(\infty)$ is independent of σ. Now let $\sigma_1\infty = \sigma_2\infty = s$. Then $\sigma_1^{-1}\sigma_2\infty = \infty$, so $\sigma_1^{-1}\sigma_2 = \pm T^n$, or $\sigma_2 = \pm\sigma_1 T^n$. Thus $\pm\sigma_2 T^m \sigma_2^{-1} = \pm\sigma_1 T^{n+m-n}\sigma_1^{-1} = \pm\sigma_1 T^m \sigma_1^{-1}$, so $\mathrm{width}(s)$ is independent of σ.

Now we show that $\mathrm{width}(s) = \mathrm{width}(gs)$ for $g \in G$. If $\sigma\infty = s$, then $g\sigma\infty = gs$. Now, $\pm g\sigma T^m \sigma^{-1} g^{-1}$ is in G iff $\pm\sigma T^m \sigma^{-1}$ is in G, so $\mathrm{width}(s) = \mathrm{width}(gs)$ as desired. In other words, width is independent of the choice of representative of the cusp.

Next, we show that $\mathrm{width}(\infty) = [\overline{\Gamma}_\infty : \overline{G}_\infty]$. Note that $\overline{\Gamma}_\infty$ is the infinite cyclic group, generated by T. It follows that $\overline{G}_\infty$ is cyclic and generated by some T^m. Note that $\overline{G}_\infty$ is not the trivial group; since G is finite index in Γ, by the pigeonhole principle there are integers a, b with $b > a > 0$ such that T^a and T^b are in the same coset, which means T^{b-a} is in G. Thus $\overline{G}_\infty$ is generated by T^m for $m > 0$; m is precisely the smallest positive integer such that $\pm T^m \in G$, so $m = \mathrm{width}(\infty)$. On the other hand, $[\langle T \rangle : \langle T^m \rangle] = m$, so $\mathrm{width}(\infty) = [\overline{\Gamma}_\infty : \overline{G}_\infty]$.

Finally, we show that $\mathrm{width}(s) = [\overline{\Gamma}_s : \overline{G}_s]$. Let $\sigma\infty = s$. Note that $\alpha s = s$ iff $\sigma^{-1}\alpha\sigma\infty = \infty$; thus $\overline{\Gamma}_s$ is infinite cyclic and generated by $T_\sigma = \sigma T \sigma^{-1}$. By the same logic as before, if m is the smallest positive integer such that $\pm T_\sigma^m = \pm\sigma T^m \sigma^{-1} \in G$, then $\overline{\Gamma}_s$ is generated by T_σ^m and $[\overline{\Gamma}_s : \overline{G}_s] = m$. In other words, $\mathrm{width}(s) = [\overline{\Gamma}_s : \overline{G}_s]$ as desired.

(b) Let $\overline{\Gamma}_\infty = \langle T \rangle$ act on $\overline{G} \setminus \overline{\Gamma}$ by right multiplication; we can also think of this action on $\{\gamma_1, \ldots, \gamma_m\}$. Of course, $\gamma_i T^n$ need not be equal to any

γ_j, but there will be a unique γ_j such that $\overline{G}\gamma_i T^n = \overline{G}\gamma_j$. Now, suppose $\gamma_1 \in \overline{G}$ represents the trivial coset. Then $\gamma_1\infty = g\infty$ for some $g \in G$, namely $g = \gamma_1$. Furthermore, for any γ_j in the $\overline{\Gamma}_\infty$-orbit of γ_1, we have $\gamma_j\infty = g\gamma_1 T^n\infty = g'\infty$ for some $g, g' \in G$. Conversely, suppose $\gamma_j\infty = g\infty$ for some $g \in G$. Then $g^{-1}\gamma_j \in \overline{\Gamma}_\infty$, say $g^{-1}\gamma_j = T^n$. Then $g^{-1}\gamma_j T^{-n} = I$ implies $\overline{G}\gamma_j T^{-n} = \overline{G}\gamma_1$, so that γ_j is in the $\overline{\Gamma}_\infty$-orbit of γ_1. Thus, the desired statement says that $[\overline{\Gamma}_\infty : \overline{G}_\infty] = |\overline{\Gamma}_\infty \cdot \gamma_1|$. By the orbit-stabilizer theorem, it suffices to show that the stabilizer of γ_1 is $\overline{G}_\infty$. Indeed, $\overline{G}\gamma_1 T^n = \overline{G}\gamma_1$, i.e. $\overline{G}T^n = \overline{G}$, is equivalent to $T^n \in \overline{G}$. Thus, the stabilizer of γ_1 is $\overline{\Gamma}_\infty \cap \overline{G} = \overline{G}_\infty$.

(c) By the proposition on slide 14 of the modular group slides, the cusps all come from coset representatives acting on ∞. In particular, we can order the coset representatives $\{\gamma_1, \dots, \gamma_m\}$ so that $s_i = \gamma_i\infty$ for $i = 1, \dots, \nu_\infty$. We will generalize part b as follows. For each $i = 1, \dots, \nu_\infty$, let S_i be the subset of $\{\gamma_1, \dots, \gamma_m\}$ consisting of γ_j such that $\gamma_j\infty = gs_i$ for some $g \in G$. Then the S_i are disjoint, since the s_i are not equivalent. We will show that $|S_i| = \text{width}(s_i)$; the desired equality immediately follows. Note that part b treats the case of the trivial coset. The argument is essentially the same; we will show that $\overline{\Gamma}_{s_i} = \langle \gamma_i T \gamma_i^{-1} \rangle = \langle T_i \rangle$ acts on $\{\gamma_1, \dots, \gamma_m\}$ such that the orbit of γ_i is S_i and the stabilizer of γ_i is $\overline{G}_{s_i}$. Indeed, let $\overline{\Gamma}_{s_i}$ act on $\overline{G} \setminus \overline{\Gamma}$ not by right multiplication, but instead by $T_i^n \cdot \overline{G}\gamma_j = \overline{G}\gamma_j T^n$. Consider also the induced action on $\{\gamma_1, \dots, \gamma_m\}$. If $\gamma_j \in \overline{\Gamma}_{s_i} \cdot \gamma_i$, then $\gamma_j\infty = g\gamma_i T^n\infty = gs_i$ for some $g \in G$. Conversely, if $\gamma_j\infty = gs_i = g\gamma_i\infty$, then $\gamma_i^{-1} g^{-1} \gamma_j = T^n$ for some n, so $\overline{G}\gamma_j = \overline{G}\gamma_i T^n$. Thus, $\overline{\Gamma}_{s_i} \cdot \gamma_i = S_i$. Next, $\overline{G}\gamma_i T^n = \overline{G}\gamma_i$ is equivalent to $\gamma_i T^n \gamma_i^{-1} = g$ for some $g \in \overline{G}$. The left hand side is T_i^n, so this says T_i^n fixes γ_i iff $T_i^n \in \overline{G} \cap \overline{\Gamma}_{s_i} = \overline{G}_{s_i}$. Thus, by the orbit-stabilizer theorem, $\text{width}(s_i) = |S_i|$, and we are done.

Now suppose $\overline{G} \lhd PSL_2(\mathbb{Z})$. Then $\sigma T^n \sigma^{-1} \in \overline{G}$ iff $T^n \in \overline{G}$, which shows that $\text{width}(\sigma\infty) = \text{width}(\infty)$ for all $\sigma \in PSL_2(\mathbb{Z})$. Since all cusps are equivalent over $PSL_2(\mathbb{Z})$, this shows that all cusps have the same width in this case.

(d) We have $\text{width}_{G_1}(s)/\text{width}_{G_2}(s) = [\overline{\Gamma}_s : \overline{G}_{1,s}]/[\overline{\Gamma}_s : \overline{G}_{2,s}] = [\overline{G}_{2,s} : \overline{G}_{1,s}]$ by basic group theory.

$\square$