MATH 7230 Homework 4

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Problem 1

For $k \geq 4$ even, define $F_k = \frac{-B_k}{2k} E_k = \sum_{n \geq 0} b_n q^n$, so that in particular $b_1 = 1$ and $b_n = \sigma_{k-1}(n)$ for $n \geq 2$. Prove that these b_n satisfy the following:

(a) For $\gcd(m, n) = 1$, we have $b_{mn} = b_m b_n$.

(b) For p prime and $r \geq 2$, we have $b_{p^r} = b_{p^{r-1}} b_p - p^{k-1} b_{p^{r-2}}$.

Proof. (a) A basic number theory fact is that if d is a positive divisor of mn where $\gcd(m, n) = 1$, then d can be uniquely written in the fg where $f, g > 0$, $f|m$, and $g|n$. This follows immediately from writing out the prime factorizations of m and n. Thus,

$$b_{mn} = \sigma_{k-1}(mn) = \sum_{d|mn} d^{k-1} = \sum_{f|m} \sum_{g|n} (fg)^{k-1}$$
$$= \sum_{f|m} f^{k-1} \sum_{g|n} g^{k-1} = \sigma_{k-1}(m)\sigma_{k-1}(n) = b_m b_n.$$

(b) The positive divisors of a prime power p^r are just $1, p, \ldots, p^j, \ldots, p^r$. Thus

$$b_{p^r} = \sigma_{k-1}(p^r) = \sum_{j=0}^{r} (p^j)^{k-1} = \sum_{j=0}^{r} (p^{k-1})^j = \frac{p^{(r+1)(k-1)} - 1}{p^{k-1} - 1},$$
$$b_{p^{r-1}} b_p - p^{k-1} b_{p^{r-2}} = (1 + p^{k-1}) \frac{p^{r(k-1)} - 1}{p^{k-1} - 1} - p^{k-1} \frac{p^{(r-1)(k-1)} - 1}{p^{k-1} - 1}.$$

To show these quantities are equal, we first multiply both quantities by $p^{k-1} - 1 \neq 0$. In particular, the second quantity gives

$$(1 + p^{k-1})(p^{r(k-1)} - 1) - p^{k-1}(p^{(r-1)(k-1)} - 1)$$
$$= \cancel{p^{r(k-1)}} - 1 + p^{(r+1)(k-1)} - \cancel{p^{k-1}} - \cancel{p^{r(k-1)}} + \cancel{p^{k-1}}$$
$$= p^{(r+1)(k-1)} - 1,$$

which is exactly $(p^{k-1} - 1) b_{p^r}$.

$\square$

Problem 2

Compute $T_2(E_4^3)$ in terms of E_4^3 and Δ.

Proof. Since E_{12} and Δ are a basis for M_{12}, we can write E_4^3 in terms of them. Furthermore, since they are eigenforms, this will make it easy to compute T_2. Clearly $a_0(E_{12}) = 1 = a_0(E_4^3)$, so $E_4^3 = E_{12} + c\Delta$ for some c. In particular, $a_1(E_4^3) = a_1(E_{12}) + a_1(c\Delta)$. We have $a_1(E_4^3) = 3a_1(E_4) = 720$, $a_1(E_{12}) = 65520/691$, and $a_1(c\Delta) = c$. Thus $c = 720 - 65520/691 = 432000/691$. I will keep using the variable c until the end. Now, $T_2(E_12) = \sigma_{11}(2)E_{12} = 2049E_{12}$ and $T_2(c\Delta) = c\tau(2)\Delta = -24c\Delta$. Thus $T_2(E_4^3) = 2049E_{12} - 24c\Delta$. But $E_{12} = E_4^3 - c\Delta$, so $T_2(E_4^3) = 2049E_4^3 - 2073c\Delta$. Miraculously, $2073 = 3 \cdot 691$, so we have

$$T_2(E_4^3) = 2049E_4^3 - 1296000\Delta.$$

$\square$

Problem 3

Compute a Hecke eigenbasis for S_{24} through the following steps:

(a) Show that $f_1 = \Delta^2$ and $f_2 = E_4^3\Delta$ form a basis for S_{24}.

(b) Our normalized eigenforms must take the form $g = f_2 + \lambda f_1$ (no coefficient on f_2 is needed because $a_1(f_2) = 1$ while $a_1(f_1) = 0$). Use this and the first few coefficients of E_4 and Δ to compute $a_2(g)$ and $a_4(g)$ in terms of λ.

(c) Use the above calculation and the Hecke relation $T_4 = T_2^2 - 2^{23}T_1$ to solve for λ.

Proof. (a) We have seen that $M_{12} = \mathbb{C}E_{12} \oplus \mathbb{C}\Delta$. In the previous problem, we showed that $E_4^3 = E_{12} + c\Delta$ for some $c \in \mathbb{C}$, so $M_{12} = \mathbb{C}E_4^3 \oplus \mathbb{C}\Delta$. We previously showed that multiplication by Δ gives an isomorphism $M_{k-12} \to S_k$; this gives the desired claim.

(b) We have

$$a_2(g) = a_2(f_2) + a_2(\lambda f_1) = \lambda - 24 + 720 = \lambda + 696.$$

To avoid a mistake in the computation of $a_4(g)$, let us write things out rather explicitly. We need $a_4(\Delta^2)$ and $a_4(E_4^3\Delta)$. Set $\delta = \Delta/q$. Then $a_4(\Delta^2) = a_2(\delta^2)$ and $a_4(E_4^3\Delta) = a_3(E_4^3\delta)$. We have

$$\delta^2 = \left(1 - 24q + 252q^2 + O(q^3)\right)^2 = 1 - 48q + (504 + 576)q^2 + O(q^3),$$

so $a_4(f_1) = 504 + 576 = 1080$. To expand f_2, we will go two series at a time, each time taking only up to order q^3. To conserve space, I will omit the $O(q^4)$. We have

$$\begin{aligned} E_4^2 &= \left(1 + 240q + 2160q^2 + 6720q^3\right)^2 \\ &= 1 + 480q + (2 * 2160 + 240^2)q^2 + (2 * 6720 + 2 * 240 * 2160)q^3 \\ &= 1 + 480q + 61920q^2 + 1050240q^3. \end{aligned}$$

Then

$$\begin{aligned} E_4^3 &= (1 + 480q + 61920q^2 + 1050240q^3)(1 + 240q + 2160q^2 + 6720q^3) \\ &= 1 + 720q + (240 * 480 + 2160 + 61920)q^2 \\ &\quad + (6720 + 2160 * 480 + 240 * 61920 + 1050240)q^3 \\ &= 1 + 720q + 179280q^2 + 16954560q^3. \end{aligned}$$

Finally,

$$\begin{aligned} a_4(f_2) &= a_3\left((1 + 720q + 179280q^2 + 16954560q^3)(1 - 24q + 252q^2 - 1472q^3)\right) \\ &= -1472 + 720 * 252 - 24 * 179280 + 16954560 = 12831808. \end{aligned}$$

Therefore, $a_4(g) = 1080\lambda + 12831808$.

(c) Applying $T_4 = T_2^2 - 2^{23}T_1$ to g gives $a_4(g) = a_2(g)^2 - 2^{23}$. In terms of λ, we have

$$\begin{gathered}1080\lambda + 12831808 = (\lambda + 696)^2 - 2^{23} = \lambda^2 + 1392\lambda - 7904192;\\ \lambda^2 + 312\lambda - 20736000 = 0;\\ (\lambda + 156)^2 = 20760336;\\ \lambda \in \{-156 + 12\sqrt{144169}, -156 - 12\sqrt{144169}\}.\end{gathered}$$

These two values of λ give us the two forms for a Hecke eigenbasis for S_{24}.

$\square$