MATH 7230 Homework 3

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Problem 1

Show that for $k \geq 2$,

$$\sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(m\tau+n)^{2k}} = \frac{1}{\zeta(2k)} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^{2k}}.$$

Proof. Since the series defining $G_{2k}(\tau)$ is absolutely convergent, we can rearrange by gcd:

$$G_{2k}(\tau) = \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (m,n)\neq(0,0)}} \frac{1}{(m\tau+n)^{2k}} = \sum_{d\geq 1} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=d}} \frac{1}{(m\tau+n)^{2k}}.$$

The set of pairs $(m,n) \in \mathbb{Z}^2$ with $\gcd(m,n) = d$ is in bijection with the set of pairs $(m,n) \in \mathbb{Z}^2$ with $\gcd(m,n) = 1$ by taking $m \mapsto dm$ and $n \mapsto dn$. Thus:

$$\begin{aligned} G_{2k}(\tau) = \sum_{d\geq 1} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(dm\tau+dn)^{2k}} &= \sum_{d\geq 1} \frac{1}{d^{2k}} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(m\tau+n)^{2k}} \\ &= \zeta(2k) \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ \gcd(m,n)=1}} \frac{1}{(m\tau+n)^{2k}}. \end{aligned}$$

Thus the claim is true. $\square$

Problem 2

Show that $M_{14}(\Gamma) = \mathbb{C}E_{14}$, and that $E_{14} = E_6E_8 = E_6E_4^2$.

Proof. By the dimension formula, $\dim M_{14}(\Gamma) = 1$. Since E_{14} is a non-zero element of $M_{14}(\Gamma)$, we must have $M_{14}(\Gamma) = \mathbb{C}E_{14}$. Now, E_6E_8 and $E_6E_4^2$ are also non-zero elements of $M_{14}(\Gamma)$, so they are scalar multiples of E_{14}. But $E_k(i\infty) = 1$ for all even $k \geq 4$, and $1 = 1 \cdot 1 = 1 \cdot 1^2$, so we must have $E_{14} = E_6E_8 = E_6E_4^2$. $\square$

Problem 3

Using the decomposition $M_k(\Gamma) = S_k(\Gamma) \oplus \mathbb{C}E_k$, show that every modular form for $\Gamma = SL_2(\mathbb{Z})$ can be expressed as a polynomial in E_4 and E_6.

Proof. We have shown explicitly that the result is true up to $k = 6$. For $k = 8$ and $k = 10$, the result is true since $E_8 = E_4^2$ and $E_{10} = E_4 E_6$, and $M_8(\Gamma) = \mathbb{C}E_8$ and $M_{10}(\Gamma) = \mathbb{C}E_{10}$. We continue by induction; assume the result is true for $k < N$. We know that $S_N(\Gamma) = \Delta M_{N-12}(\Gamma)$, and Δ is a polynomial in E_4 and E_6 by definition, so elements of $S_N(\Gamma)$ are polynomials in E_4 and E_6 by the inductive hypothesis. Furthermore, $E_N = E_6^2 E_{N-12}$, since both sides are non-zero non-cusp modular forms of weight k, the space of such forms is one-dimensional, and both sides have the same value of 1 at $i\infty$. By induction, E_{N-12} is a polynomial in E_4 and E_6, which implies $E_6^2 E_{N-12}$ is as well. Since $S_N(\Gamma)$ and $\mathbb{C}E_N$ consist of polynomials in E_4 and E_6, we have $M_N(\Gamma)$ consists of polynomials in E_4 and E_6. $\square$

Problem 4

Let

$$D = \frac{1}{2\pi i}\frac{d}{d\tau} = q\frac{d}{dq}.$$

(a) Using the transformation property for E_2, show that

$$D(E_2) - \frac{1}{12}E_2^2 = -\frac{1}{12}E_4.$$

(b) Deduce the Ramanujan formula

$$DE_2 = \frac{E_2^2 - E_4}{12}.$$

(c) Show similarly that

$$DE_4 = \frac{E_2E_4 - E_6}{3},$$
$$DE_6 = \frac{E_2E_6 - E_4^2}{2}.$$

Proof. (a) Call the expression on the left-hand side L. We will show that $L \in M_4$. Thus, we will show $L = L|_4\gamma$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. We have

$$L|_4\gamma = D(E_2)|_4\gamma - \frac{1}{12}E_2^2|_4\gamma.$$

Using the lemma from lecture that concerns D and slash operator, we have

$$D(E_2)|_4\gamma = D(E_2|_2\gamma) + \frac{2}{2\pi i}\frac{c}{c\tau + d}E_2|_2\gamma.$$

Now, note that $f^2|_{2k}\gamma = (f|_k\gamma)^2$, since unraveling the definition on both sides gives

$$f^2(\tau)(c\tau + d)^{-2k} = (f(\tau)(c\tau + d)^{-k})^2.$$

Thus

$$-\frac{1}{12}E_2^2|_4\gamma = -\frac{1}{12}(E_2|_2\gamma)^2.$$

In total, we have

$$L|_4\gamma = D(E_2|_2\gamma) - \frac{ic}{\pi(c\tau + d)}E_2|_2\gamma - \frac{1}{12}(E_2|_2\gamma)^2.$$

Now, we know $E_2|_2\gamma = E_2 - \dfrac{6ic}{\pi(c\tau+d)}$. We substitute this in and do some algebra:

$$
\begin{aligned}
L|_4\gamma &= D(E_2) - \frac{6ic}{\pi} D\left(\frac{1}{c\tau+d}\right) - \frac{ic}{\pi(c\tau+d)}E_2 - \frac{6c^2}{\pi^2(c\tau+d)^2} - \frac{1}{12}\left(E_2 - \frac{6ic}{\pi(c\tau+d)}\right)^2 \\
&= D(E_2) + \frac{6ic}{\pi}\frac{1}{2\pi i}\frac{c}{(c\tau+d)^2} - \frac{ic}{\pi(c\tau+d)}E_2 - \frac{6c^2}{\pi^2(c\tau+d)^2} - \frac{1}{12}\left(E_2 - \frac{6ic}{\pi(c\tau+d)}\right)^2 \\
&= D(E_2) - \cancel{\frac{3c^2}{\pi^2(c\tau+d)^2}} - \cancel{\frac{ic}{\pi(c\tau+d)}E_2} - \frac{1}{12}E_2^2 + \cancel{\frac{ic}{\pi(c\tau+d)}E_2} + \cancel{\frac{3c^2}{\pi^2(c\tau+d)^2}} \\
&= D(E_2) - \frac{1}{12}E_2^2 = L.
\end{aligned}
$$

Finally, L is holomorphic because E_2 and thus $D(E_2)$ are holomorphic. Thus, $L \in M_4$. Of course, the right hand side is in $M_4 = \mathbb{C}E_4$, and is the unique element of M_4 whose constant Fourier coefficient is $-1/12$. It then suffices to look at the constant Fourier coefficient of the left hand side. The $D(E_2)$ term will not contribute, since it multiplies a power series with no negative powers of q by q. Since the constant term of E_2 is 1, it follows that the constant term of L is $-\frac{1}{12} \cdot 1^2 = -1/12$. This completes the proof.

(b) Add $E_2^2/12$ to both sides of the equation.

(c) We proceed similarly. Since $M_6 = \mathbb{C}E_6$ and $M_8 = \mathbb{C}E_8$, and $E_4^2 = E_8$, it suffices to show that $DE_4 - E_2E_4/3 \in M_6$ and $DE_6 - E_2E_6/2 \in M_8$, and that their constant Fourier coefficients are $-1/3$ and $-1/2$, respectively. Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$. Then

$$
\begin{aligned}
D(E_4)|_6\gamma &= D(E_4|_4\gamma) + \frac{4}{2\pi i}\frac{c}{c\tau+d}E_4|_4\gamma \\
&= D(E_4) - \frac{2ic}{\pi(c\tau+d)}E_4,
\end{aligned}
$$

and

$$
\begin{aligned}
(E_2E_4)|_6\gamma &= E_2|_2\gamma E_4|_4\gamma = E_2|_2\gamma E_4 \\
= \left(E_2 - \frac{6ic}{\pi(c\tau+d)}\right)E_4 &= E_2E_4 - \frac{6ic}{\pi(c\tau+d)}E_4.
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\left(DE_4 - \frac{1}{3}E_2E_4\right)|_6\gamma &= DE_4 - \frac{2ic}{\pi(c\tau+d)}E_4 - \frac{1}{3}E_2E_4 + \frac{2ic}{\pi(c\tau+d)}E_4 \\
&= DE_4 - \frac{1}{3}E_2E_4.
\end{aligned}
$$

Since E_2, E_4 are holomorphic, so is $DE_4 - E_2E_4/3$. To conclude $DE_4 - E_2E_4/3 = -E_6/3$, we must show that the constant Fourier coefficient of $DE_4 - E_2E_4/3$ is $-1/3$. As in part (a), DE_4 contributes nothing to the constant coefficient, while the constant coefficients of E_2 and E_4 are both one, so the constant coefficient of $-E_2E_4/3$ is indeed $-1/3$.

Now we do the same thing for $DE_6 - E_2E_6/2$.

$$\begin{aligned} D(E_6)|_8\gamma &= D(E_6|_6\gamma) + \frac{6}{2\pi i}\frac{c}{c\tau + d}E_6|_6\gamma \\ &= D(E_6) - \frac{3ic}{\pi(c\tau + d)}E_6, \end{aligned}$$

and

$$\begin{aligned} (E_2E_6)|_8\gamma = E_2|_2\gamma E_6|_6\gamma = E_2|_2\gamma E_6 \\ = \left(E_2 - \frac{6ic}{\pi(c\tau + d)}\right)E_6 = E_2E_6 - \frac{6ic}{\pi(c\tau + d)}E_6. \end{aligned}$$

Thus

$$\begin{aligned} \left(DE_6 - \frac{1}{2}E_2E_6\right)|_8\gamma &= DE_6 - \frac{3ic}{\pi(c\tau + d)}E_6 - \frac{1}{2}E_2E_4 + \frac{3ic}{\pi(c\tau + d)}E_6 \\ &= DE_6 - \frac{1}{2}E_2E_6. \end{aligned}$$

Once again, this is holomorphic, so it is in M_8. As before, DE_6 has 0 constant Fourier coefficient, whereas the constant Fourier coefficients of E_2 and E_6 are both 1; thus $DE_6 - E_2E_6/2$ has a constant Fourier coefficient of $-1/2$, meaning it is equal to $-E_8/2 = -E_4^2/2$.

$\square$

Problem 5

Let $\tau(n)$ be the nth coefficient of the Fourier expansion of Δ. Prove that $\tau(n) \equiv \sigma_{11}(n) \bmod 691$ by showing that

$$\frac{691}{65520}E_{12} - \Delta = \frac{691}{112320}E_4^3 + \frac{691}{157248}E_6^2.$$

Proof. First, assume that the above equation is true. Since E_4 and E_6 have integer coefficients in their Fourier expansions, and 112320 and 157248 are invertible mod 691, it follows that the right hand side of the equation has each Fourier coefficient equivalent to 0 mod 691. On the other hand, using $B_{12} = \frac{-691}{2730}$, we have

$$\begin{aligned}\frac{691}{65520}E_{12}(q) &= \frac{691}{65520} - \frac{691}{65520}(24)\left(\frac{-2730}{691}\right)\left(\sum_{n=1}^{\infty}\sigma_{11}(n)q^n\right)\\ &= \frac{691}{65520} + \sum_{n=1}^{\infty}\sigma_{11}(n)q^n.\end{aligned}$$

The constant coefficient vanishes mod 691, which aligns with the fact that $\tau(0) = 0$. For $n > 0$, we get $\sigma_{11}(n) - \tau(n)$ on the left hand side of the equation in the problem statement, and as we said earlier, 0 mod 691 on the right hand side. Thus, the congruence is true.

Now, to prove the relation between E_4, E_6, E_{12}, and Δ, it suffices to show that the constant and q coefficients of the left and right hand side agree. Indeed, since $M_{12} = S_{12} \oplus \mathbb{C}E_{12} = \mathbb{C}\Delta \oplus \mathbb{C}E_{12}$, if $f \in M_{12}$ has $f(q) = a + bq + O(q^2)$, then $f = aE_{12} + (b - ac)\Delta$, where c is the q coefficient of $E_{12}(q)$. The constant coefficient of the left hand side is $\frac{691}{65520} - 0$, and the constant coefficient of the right hand side is $\frac{691}{112320} + \frac{691}{157248}$. One can check explicitly that these numbers are equal. The q coefficient of the left hand side is $\sigma_{11}(1) - \tau(1) = 1 - 1 = 0$, and the q coefficient of the right hand side is

$$\frac{691}{112320}(3)\left(\frac{-8}{-1/30}\right) + \frac{691}{157248}(2)\left(\frac{-12}{1/42}\right) = \frac{691}{156} - \frac{691}{156} = 0.$$

Note that we have used the fact that $(1+aq+O(q^2))^n = 1+naq+O(q^2)$, which explains the appearance of the factors of 3 and 2. We have also used the explicit values of B_4 and B_6. Since the left and right hand side have equal constant and q coefficients of their Fourier expansions, they are equal, so we are done. $\square$