MATH 7230 Homework 2

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Problem 1

For any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{R})$, define the function $j_\gamma(\tau) = c\tau + d$. Show that

(a) $j_{\gamma_1\gamma_2}(\tau) = j_{\gamma_1}(\gamma_2\tau)j_{\gamma_2}(\tau)$.

(b) $\gamma\tau - \gamma z = (\tau - z)j_\gamma(\tau)^{-1}j_\gamma(z)^{-1}$.

(c) $d(\gamma\tau) = j_\gamma(\tau)^{-2}d\tau$.

(d) $|j_\gamma(\tau)|^2\mathrm{Im}(\gamma\tau) = \mathrm{Im}(\tau)$.

Proof. (a) We expand both sides. Let $\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}$. Then

$$\gamma_1\gamma_2 = \begin{pmatrix} - & - \\ a_2c_1 + c_2d_1 & b_2c_1 + d_1d_2 \end{pmatrix},$$

and in particular,

$$j_{\gamma_1\gamma_2}(\tau) = (a_2c_1 + c_2d_1)\tau + b_2c_1 + d_1d_2.$$

On the other hand,

$$\begin{aligned} j_{\gamma_1}(\gamma_2\tau)j_{\gamma_2}(\tau) &= (c_1\gamma_2\tau + d_1)(c_2\tau + d_2) \\ &= \left(c_1\frac{a_2\tau + b_2}{c_2\tau + d_2} + d_1\right)(c_2\tau + d_2) \\ &= c_1(a_2\tau + b_2) + d_1(c_2\tau + d_2) \\ &= (a_2c_1 + c_2d_1)\tau + b_2c_1 + d_1d_2 = j_{\gamma_1\gamma_2}(\tau). \end{aligned}$$

(b) Let us compare both sides in more detail. The claim is that

$$\frac{a\tau + b}{c\tau + d} - \frac{az + b}{cz + d} = \frac{\tau - z}{(c\tau + d)(cz + d)}.$$

Equivalently, after multiplying both sides by $j_\gamma(\tau)j_\gamma(z)$, the claim is that

$$(a\tau + b)(cz + d) - (c\tau + d)(az + b) = \tau - z.$$

Expanding and simplifying the left hand side gives

$$\begin{aligned} &ac\tau z + ad\tau + bcz + bd - ac\tau z - bc\tau - adz - bd \\ &= (ad - bc)\tau - (ad - bc)z. \end{aligned}$$

Since $\gamma \in SL_2(\mathbb{R})$, we have $ad - bc = 1$, so the claim is true.

(c) We have

$$\begin{aligned} \frac{d}{d\tau}\left(\frac{a\tau + b}{c\tau + d}\right) &= \frac{a}{c\tau + d} - \frac{c(a\tau + b)}{(c\tau + d)^2} \\ &= \frac{ac\tau + ad - ac\tau - bc}{(c\tau + d)^2} = \frac{ad - bc}{(c\tau + d)^2}. \end{aligned}$$

Once again, since $\gamma \in SL_2(\mathbb{R})$, we have $ad - bc = 1$, so we get the desired claim.

(d) Write $\tau = x + yi$. Then

$$\gamma\tau = \frac{ax + ayi + b}{cx + cyi + d} = \frac{(ax + b + ayi)(cx + d - cyi)}{|j_\gamma(\tau)|^2},$$

so

$$\begin{aligned} \mathrm{Im}(\gamma\tau) &= \frac{(ax + b)(-cy) + (ay)(cx + d)}{|j_\gamma(\tau)|^2} \\ &= \frac{-acxy - bcy + acxy + ady}{|j_\gamma(\tau)|^2} = \frac{(ad - bc)\mathrm{Im}(\tau)}{|j_\gamma(\tau)|^2}. \end{aligned}$$

Since $ad - bc = 1$, we have the desired claim.

□

Problem 2

For any point $\tau \in \mathbb{H}$, write $\tau = x + iy$ with $x, y \in R$ and $y > 0$. Show that the differential form $ds^2 = (dx^2 + dy^2)/y^2$ is invariant under the action of $SL_2(\mathbb{R})$.

Proof. Note that we can write $ds^2 = d\tau d\overline{\tau}/y^2$. For $\gamma \in SL_2(\mathbb{R})$, we have $\overline{\gamma\tau} = \gamma\overline{\tau}$, since the entries of γ are real. Using parts (c) and (d) of Problem 1, we have

$$\begin{aligned} d(\gamma s)^2 = \frac{d(\gamma\tau)d(\gamma\overline{\tau})}{\operatorname{Im}(\gamma\tau)^2} &= \frac{j_\gamma(\tau)^{-2}d\tau j_\gamma(\overline{\tau})^{-2}d\overline{\tau}}{\operatorname{Im}(\tau)^2|j_\gamma(\tau)|^{-4}} \\ &= \frac{|j_\gamma(\tau)|^4}{j_\gamma(\tau)^2 j_\gamma(\overline{\tau})^2} ds^2. \end{aligned}$$

We are reduced to showing $|j_\gamma(\tau)|^4 = j_\gamma(\tau)^2 j_\gamma(\overline{\tau})^2$. This is clear, since $j_\gamma(\overline{\tau}) = c\overline{\tau} + d = \overline{c\tau + d} = \overline{j_\gamma(\tau)}$. □

Problem 3

Show that every elliptic point of order 3 of $PSL_2(\mathbb{Z}) \setminus \mathbb{H}$ is equivalent to $e^{\pi i/3}$.

Proof. Assume for now the claim in the notes that every order 3 elliptic element of $PSL_2(\mathbb{Z})$ is conjugate to either $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Note this basic fact: If x is fixed by MLM^{-1}, then $M^{-1}x$ is fixed by L. Indeed, $LM^{-1}x = M^{-1}MLM^{-1}x = M^{-1}x$. In other words, the fixed points of conjugate elements are in the same orbit. We can therefore study the fixed points of $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ and $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$ and show that they are equivalent to $\rho := e^{\pi i/3}$.

Case 1: $\tau \in \mathbb{H}$ is fixed by $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$, so $\tau = \dfrac{-1}{\tau - 1}$. Rearranging gives $\tau^2 - \tau + 1 = 0$, which has one solution in $\mathbb{H}$, namely ρ.

Case 2: $\tau \in \mathbb{H}$ is fixed by $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\tau = \dfrac{-\tau + 1}{-\tau}$. Rearranging give $\tau^2 - \tau + 1 = 0$ again, so $\tau = \rho$.

We now show that every order 3 elliptic element of $PSL_2(\mathbb{Z})$ is conjugate to either $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$ or $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. We know that an elliptic element must have trace with absolute value less than 2. Since the trace of an integer matrix is an integer, the only possibilities are $-1, 0, 1$. The trace zero case gives order 2:

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & 0 \\ 0 & a^2 + bc \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

By multiplying by $-\mathrm{id}$, we can assume that the trace of our matrix is -1 (and we work in $SL_2(\mathbb{Z})$ from now on). Thus, we start with some matrix $\begin{pmatrix} a & b \\ c & -a-1 \end{pmatrix}$. Notice that if $b = 0$, then we would have $1 = -a^2 - a$ as the determinant, but the roots of $x^2 + x + 1$ are not integers. Thus $b \neq 0$. We claim that if $b > 0$, then our matrix is conjugate to $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$, and if $b < 0$, then our matrix is conjugate to $\begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}$. I will assume $b > 0$ and withhold the proof for $b < 0$, since it is very similar (but just as long).

The proof is long, so let me start with an outline. We will exploit the fact that $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generate $SL_2(\mathbb{Z})$. In particular, to show that two matrices are conjugates in $SL_2(\mathbb{Z})$, we can apply sequences of conjugation by $S^{\pm 1}$ or $T^{\pm 1}$. We will first show that all matrices $\begin{pmatrix} a & 1 \\ c & -a-1 \end{pmatrix} \in SL_2(\mathbb{Z})$ are conjugate to each other, and then show that for $a = 0$, $\begin{pmatrix} 0 & 1 \\ c & -1 \end{pmatrix}$ is conjugate to $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Then, we will strongly induct on b; if we start with some matrix $\begin{pmatrix} a & b \\ c & -a-1 \end{pmatrix}$ with $b > 1$, we can find some $\begin{pmatrix} a' & b' \\ c' & -a'-1 \end{pmatrix}$ conjugate to it with $0 < b' < b$, and by induction, this matrix will be conjugate to $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

If $\begin{pmatrix} a & 1 \\ c & -a-1 \end{pmatrix} \in SL_2(\mathbb{Z})$, then $c = -a^2 - a - 1$, so the matrix is determined by a; let's call it M_a. We will show that M_a is conjugate to M_{a+1} and M_{a-1}, which will imply that it is conjugate to $M_0 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$. Indeed, it can be verified that

$$M_{a+1} = STSM_aS^{-1}T^{-1}S^{-1},$$
$$M_{a-1} = ST^{-1}SM_aS^{-1}TS^{-1}.$$

Thus any M_a is conjugate to M_0. Furthermore, one can verify that $SM_0S^{-1} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$. Thus, all matrices $\begin{pmatrix} a & 1 \\ c & -a-1 \end{pmatrix}$ in $SL_2(\mathbb{Z})$ are conjugate to $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

Now consider $M = \begin{pmatrix} a & b \\ c & -a-1 \end{pmatrix}$ with $b > 1$. We will show that it is conjugate to some $\begin{pmatrix} a' & b' \\ c' & -a'-1 \end{pmatrix}$ with $0 < b' < b$. More precisely, we will show that there is an integer n such that $T^{-1}ST^nSMS^{-1}T^{-n}S^{-1}T$ has its top right entry positive and less than b.

We will work backwards; let us consider what conditions on M would imply that $T^{-1}MT$ has its top right entry positive and less than b. We have

$$T^{-1}MT = \begin{pmatrix} * & 2a+b-c+1 \\ * & * \end{pmatrix},$$

so we want $0 < 2a + b - c + 1 < b$. Since $\det M = -a^2 - a - bc = 1$, we can rewrite the first inequality as

$$0 < 2a + b + \frac{a^2+a+1}{b} + 1.$$

Since b is positive, we can multiply both sides by b to get

$$0 < 2ab + b^2 + a^2 + a + 1 + b = (a+b)^2 + (a+b) + 1.$$

This is true because $x^2 + x + 1 > 0$ for all real x. The second inequality can be rewritten as

$$2a + \frac{a^2+a+1}{b} + 1 < 0.$$

Since $a^2+a+1>0$ and $b>0$, this is equivalent to

$$\frac{b(2a+1)}{a^2+a+1} < -1.$$

This inequality is not always true, as can be shown by choosing $a=-8, b=3$. However, we will show that M is conjugate to some matrix whose top left and top right entries (as a and b, resp.) satisfy this inequality. We have

$$ST^nSMS^{-1}T^{-n}S^{-1} = \begin{pmatrix} a+nb & b \\ * & * \end{pmatrix}.$$

We will show that there exists an integer n such that

$$\frac{b(2(a+nb)+1)}{(a+nb)^2+(a+nb)+1} < -1.$$

From our work above, this will imply that the top right entry of $T^{-1}ST^nSMS^{-1}T^{-n}S^{-1}T$ is positive and less than b. Now, we rearrange the above inequality, again using that $x^2+x+1>0$ for real x, and in particular for $x=a+nb$:

$$b^2n^2+(2b^2+2ab+b)n+(2ab+b+a^2+a+1)<0.$$

If we can show that the roots of the quadratic $b^2x^2+(2b^2+2ab+b)x+(2ab+b+a^2+a+1)$ are real and are a distance greater than 1 apart, we will be done; any open interval of length greater than 1 contains an integer, so we can choose n to be an integer between the roots. The difference of the roots is

$$\frac{\sqrt{(2b^2+2ab+b)^2-4b^2(2ab+b+a^2+a+1)}}{b^2}.$$

The inside of the square root simplifies tremendously, giving $4b^4-3b^2$. Since $b>1$, we have $b^2>0$ and $4b^2-3>0$ since both x^2 and $4x^2-3$ are increasing functions and positive at $b=1$. Thus, the roots are real. It remains to show

$$\frac{\sqrt{4b^4-3b^2}}{b^2} > 1.$$

Since everything in sight is positive, we can safely square both sides and rearrange to get $3b^2-3>0$, which is true since $b>1$.

In case it isn't clear, let me also prove the claim that an open interval of length greater than 1 contains an integer. Say the open interval is (α,β), and that $\beta-\alpha>1$. If α is not an integer, then we can take $n=\lceil\alpha\rceil$. Indeed, by definition, $\alpha\le n$, but it can't be equal since $\alpha\notin\mathbb{Z}$, and $n<\alpha+1<\beta$, since otherwise $n-1$ would be a smaller integer $\ge\alpha$, contradicting the definition of n. If α is an integer, then $\alpha+1$ is an integer in (α,β).

To recap:

1. We began by showing that an elliptic element of order 3 in $PSL_2(\mathbb{Z})$ must have trace ± 1.

2. We fixed the trace to be -1 and worked strictly in $SL_2(\mathbb{Z})$.

3. The top right entry of such an element must be non-zero. We assumed that it is positive, since the negative case is similar (but equally as long).

4. We showed that all matrices $\begin{pmatrix} a & 1 \\ c & -a-1 \end{pmatrix} \in SL_2(\mathbb{Z})$ are conjugate to each other.

5. We showed that $\begin{pmatrix} 0 & 1 \\ c & -1 \end{pmatrix} \in SL_2(\mathbb{Z})$ is conjugate to $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

6. We showed that $\begin{pmatrix} a & b \\ c & -a-1 \end{pmatrix} \in SL_2(\mathbb{Z})$ for $b > 1$ is conjugate to some $\begin{pmatrix} a' & b' \\ c' & -a'-1 \end{pmatrix} \in SL_2(\mathbb{Z})$ with $0 < b' < b$.

7. By strong induction on b, we have that all $\begin{pmatrix} a & b \\ c & -a-1 \end{pmatrix} \in SL_2(\mathbb{Z})$ with $b > 0$ are conjugate to $\begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}$.

$\square$

Problem 4

Take $\gamma = \begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix}$. Write γ as a product of $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Proof. Recall that $T^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ for all $n \in \mathbb{Z}$. We will use this and multiplication by S to reduce γ down.

$$\begin{aligned}
\gamma_1 &:= \gamma T^{-2} = \begin{pmatrix} 4 & 9 \\ 11 & 25 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix}, \\
\gamma_2 &:= \gamma_1 S = \begin{pmatrix} 4 & 1 \\ 11 & 3 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -4 \\ 3 & -11 \end{pmatrix}, \\
\gamma_3 &:= \gamma_2 T^4 = \begin{pmatrix} 1 & -4 \\ 3 & -11 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}, \\
\gamma_4 &:= \gamma_3 S = \begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix}, \\
\gamma_5 &:= \gamma_4 T^3 = \begin{pmatrix} 0 & -1 \\ 1 & -3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = S.
\end{aligned}$$

Working backwards, we find $\gamma T^{-2} S T^4 S T^3 = S$, so $\gamma = S T^{-3} S^{-1} T^{-4} S^{-1} T^2$. $\square$