MATH 7230 Homework 1

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Problem 1

Given $a \in \mathbb{C}$, let D_a be the parallelogram with vertices $a, a+\omega_2, a+\omega_1+\omega_2, a+\omega_1$. Prove that D_a is a fundamental domain for the group G of translations along the lattice Λ generated by ω_1, ω_2. That is,

(i) Given any $z \in \mathbb{C}$, there exists $\omega \in \Lambda$ such that $z + \omega \in D_a$.

(ii) If there are two points z_1, z_2 in the interior of D_a such that $z_1 = z_2 + \omega$ for some $\omega \in \Lambda$, then $\omega = 0$.

Proof. (i) Since ω_1, ω_2 are two $\mathbb{R}$-linearly independent complex numbers and $\mathbb{C}$ has $\mathbb{R}$-dimension 2, they form an $\mathbb{R}$-basis for $\mathbb{C}$. Thus we can write $z - a = t_1\omega_1 + t_2\omega_2$ for $t_1, t_2 \in \mathbb{R}$. Furthermore, write $t_1 = b + t_1'$ and $t_2 = c + t_2'$ with $b, c \in \mathbb{Z}$ and $t_1', t_2' \in [0, 1)$. Let $\omega = -b\omega_1 - c\omega_2$; this is a point in Λ. Then $z + \omega = a + t_1'\omega_1 + \omega_2'$. D_a can be explicitly described as the set of points $a + t_1\omega_1 + t_2\omega_2$ with $t_1, t_2 \in [0, 1]$. Thus, $z + \omega \in D_a$ as desired.

(ii) The interior of D_a is the set of points $a+t_1\omega_1+t_2\omega_2$ with $t_1, t_2 \in (0, 1)$. Let z_1 and $z_2 = a + t_1\omega_1 + t_2\omega_2$ be in the interior of D_a. Suppose $z_1 = z_2 + \omega$ for $\omega = b\omega_1 + c\omega_2 \in \Lambda$. Then $z_1 = a + (t_1 + b)\omega_1 + (t_2 + c)\omega_2$ is in D_a. This means $t_1 + b$ and $t_2 + c$ are in $(0, 1)$. Note that if $x, y \in (0, 1)$, then $x - y \in (-1, 1)$. In particular, this implies $b, c \in (-1, 1)$. But the only integer contained in $(-1, 1)$ is 0, so $b = c = 0$, so $\omega = 0$. Thus $z_1 = z_2 + 0 = z_2$.

□

Problem 2

Let Λ be a lattice and let $\wp$ be its Weierstrass $\wp$-function.

(a) Show that $\wp$ is even and that $\wp'$ is doubly-periodic with respect to Λ.

(b) For $i = 1, 2$ show that the function $\wp(z + \omega_i) - \wp(z)$ is a constant c_i by taking its derivative. Substitute $z = -\omega_i/2$ to show $c_i = 0$. Conclude that $\wp$ is doubly-periodic with respect to Λ.

Proof. (a) Note that there is a fixed-point free involution on $\Lambda \setminus \{0\}$ given by $\omega \mapsto -\omega$. Since the summation in the definition of $\wp$ is indexed over $\Lambda \setminus \{0\}$ and is absolutely convergent, we can swap ω and $-\omega$ to get the exact same sum. Therefore:

$$\begin{aligned}
\wp(-z) &= \frac{1}{(-z)^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(-z-\omega)^2} - \frac{1}{\omega^2} \right) \\
&= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-(-\omega))^2} - \frac{1}{(-\omega)^2} \right) \\
&= \frac{1}{z^2} + \sum_{\omega\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right) = \wp(z).
\end{aligned}$$

Similarly, there is a fixed-point free involution on Λ given by $\omega \mapsto \omega - \omega'$ for some $\omega' \in \Lambda$. Since the summation over Λ in $\wp'$ is absolutely convergent, we can swap ω and $\omega - \omega'$ for any $\omega' \in \Lambda$ to get the same sum. Therefore, for any $\omega' \in \Lambda$:

$$\begin{aligned}
\wp'(z + \omega') &= -2 \sum_{\omega\in\Lambda} \frac{1}{(z + \omega' - \omega)^3} = -2 \sum_{\omega\in\Lambda} \frac{1}{(z - (\omega - \omega'))^3} \\
&= -2 \sum_{\omega\in\Lambda} \frac{1}{(z-\omega)^3} = \wp'(z).
\end{aligned}$$

(b) Let $f_i(z) = \wp(z+\omega_i) - \wp(z)$. Then $f_i'(z) = \wp'(z+\omega_i) - \wp'(z) = 0$ by part (a). Thus $f_i(z) = c_i$ is a constant. Since ω_1, ω_2 are a basis for $\mathbb{C}$ over $\mathbb{R}$, the point $z = -\omega_i/2$ is not in Λ since it is a non-integer linear combination of ω_1, ω_2. Thus, we can evaluate f_i at $z = -\omega_i/2$ to get $c_i = \wp(\omega_i/2) - \wp(-\omega_i/2) = 0$ by part (a). Thus, $\wp(z + \omega_i) = \wp(z)$ for $i = 1, 2$, so $\wp$ is doubly-periodic with respect to Λ as desired.

$\square$

Problem 3

Set $\wp(z;\tau) = \wp(z;\tau\mathbb{Z}\oplus\mathbb{Z})$, so $\wp(z;\Lambda) = \omega_2^{-2}\wp(z/\omega_2;\omega_1/\omega_2)$. For all $z \in \mathbb{C}, \tau \in \mathbb{H}$, and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, show that

$$\wp\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^2\wp(z;\tau).$$

Proof. Using $\wp(z;\Lambda) = \omega_2^{-2}\wp(z/\omega_2;\omega_1/\omega_2)$, we see that

$$(c\tau+d)^{-2}\wp\left(\frac{z}{c\tau+d};\frac{a\tau+b}{c\tau+d}\right) = \wp(z;\Lambda')$$

where $\Lambda' = (a\tau+b)\mathbb{Z}\oplus(c\tau+d)\mathbb{Z}$. Let $\Lambda = \tau\mathbb{Z}\oplus\mathbb{Z}$. If we can show $\Lambda = \Lambda'$, then we will be done. Clearly $\Lambda' \subset \Lambda$, since $e(a\tau+b)+f(c\tau+d) = (ae+cf)\tau+(be+df)$. On the other hand, since $ad-bc = 1$, we have

$$\begin{aligned} 1 &= -c(a\tau+b)+a(c\tau+d) \in \Lambda', \\ \tau &= d(a\tau+b)-b(c\tau+d) \in \Lambda'. \end{aligned}$$

Thus $\Lambda \subset \Lambda'$. $\square$

Problem 4

Consider two lattices $\Lambda = \omega_1\mathbb{Z} \oplus \omega_2\mathbb{Z}$ and $\Lambda' = \omega_1'\mathbb{Z} \oplus \omega_2'\mathbb{Z}$ and with $\omega_1/\omega_2, \omega_1'/\omega_2' \in \mathbb{H}$. Show that $\Lambda' = \Lambda$ iff

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

Proof. First, suppose $\Lambda' = \Lambda$. This can be split into two statements: $\Lambda' \subset \Lambda$ and $\Lambda \subset \Lambda'$. From the first case, we see that $\omega_1', \omega_2' \in \Lambda$, meaning by definition that there are integers a, b, c, d such that $\omega_1' = a\omega_1 + b\omega_2$ and $\omega_2' = c\omega_1 + d\omega_2$, or alternatively,

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

From the second containment we have $\omega_1, \omega_2 \in \Lambda'$, which means there are integers a', b', c', d' such that $\omega_1 = a'\omega_1' + b'\omega_2'$ and $\omega_2 = c'\omega_1' + d'\omega_2'$, or alternatively,

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

We then have

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} w & x \\ y & z \end{pmatrix}$. Since ω_1', ω_2' form an $\mathbb{R}$-basis for $\mathbb{C}$, the equalities $(w-1)\omega_1' + x\omega_2' = 0$ and $y\omega_1' + (z-1)\omega_2' = 0$ imply that $\begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. A similar argument shows $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, so $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$. Since the determinant is multiplicative, we know $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a unit in $\mathbb{Z}$, so either -1 or 1. It remains to show that it is in fact 1. To do so, we use the fact that $\omega_1/\omega_2, \omega_1'/\omega_2' \in \mathbb{H}$. Write $\tau = \omega_1/\omega_2 = \alpha + \beta i$. Then

$$\begin{aligned} \frac{\omega_1'}{\omega_2'} = \frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2} = \frac{a\tau + b}{c\tau + d} = \frac{a\alpha + b + a\beta i}{c\alpha + d + c\beta i} \\ = \frac{(a\alpha + b + a\beta i)(c\alpha + d - c\beta i)}{|c\tau + d|^2}. \end{aligned}$$

The denominator is positive. The imaginary part of the numerator is $a\beta(c\alpha + d) - c\beta(a\alpha + b) = (ac - bd)\beta$. We know that $\beta > 0$ and $(ac - bd)\beta > 0$, so we must have $ac - bd > 0$. Since $|ac - bd| = 1$, we have $ac - bd = 1$ as desired.

Now suppose

$$\begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \text{ for some } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}).$$

For all integers m, n, we have $m\omega_1' + n\omega_2' = (am + cn)\omega_1 + (bm + dn)\omega_2 \in \Lambda$, so $\Lambda' \subset \Lambda$. Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, its inverse is also an integer matrix, call it $\begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$. Then

$$\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} \omega_1' \\ \omega_2' \end{pmatrix}.$$

For all integers m, n, we have $m\omega_1 + n\omega_2 = (a'm + c'n)\omega_1' + (b'm + d'n)\omega_2' \in \Lambda'$, so $\Lambda \subset \Lambda'$. Thus $\Lambda = \Lambda'$ as desired. $\square$