MATH 7230 Homework 9

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1 Problem 8.1 1

Let $D(Q)$ be the decomposition group of the prime ideal Q. It follows from the definition of stabilizer subgroup that $D(\sigma(Q)) = \sigma D(Q)\sigma^{-1}$ for every $\sigma \in G$. Show that the inertia subgroup also behaves in this manner, that is, $I(\sigma(Q)) = \sigma I(Q)\sigma^{-1}$.

Proof. Let $\sigma' \in \sigma D(Q)\sigma^{-1}$, so $\sigma' = \sigma\sigma''\sigma^{-1}$ with $\sigma'' \in D(Q)$. If $x \in \sigma(Q)$, then $\sigma^{-1}(x) \in \sigma^{-1}\sigma(Q) = Q$. Then $\sigma''\sigma^{-1}(x) \in Q$, since $\sigma''(Q) = Q$. Then $\sigma\sigma''\sigma^{-1}(x) \in \sigma(Q)$. Thus $\sigma'(x) \in \sigma(Q)$ for all $x \in \sigma(Q)$, so $\sigma' \in D(\sigma(Q))$.

Now let $\sigma' \in D(\sigma(Q))$. Let $\sigma'' = \sigma^{-1}\sigma'\sigma$. Let $x \in Q$. Then $\sigma(x) \in \sigma(Q)$. $\sigma'\sigma(x) \in \sigma(Q)$, since $\sigma' \in D(\sigma(Q))$. Then $\sigma^{-1}\sigma'\sigma(x) \in \sigma^{-1}\sigma(Q) = Q$. Thus $\sigma''(x) \in Q$ for all $x \in Q$, so $\sigma'' \in D(Q)$. Thus $\sigma' = \sigma\sigma''\sigma^{-1} \in \sigma D(Q)\sigma^{-1}$.

These two results imply $D(\sigma(Q)) = \sigma D(Q)\sigma^{-1}$.

Now, let $\sigma' = \sigma\sigma''\sigma^{-1} \in \sigma I(Q)\sigma^{-1}$. As $\sigma'' \in I(Q)$, we have $\sigma''(x) - x \in Q$ for all $x \in B$. Then $\sigma''\sigma^{-1}(x) - \sigma^{-1}(x) \in Q$, so that $\sigma(\sigma''\sigma^{-1}(x) - \sigma^{-1}(x)) = \sigma\sigma''\sigma^{-1}(x) - x \in \sigma(Q)$.Thus $\sigma'(x) - x \in \sigma(Q)$ for all $x \in B$, so $\sigma' \in I(\sigma(Q))$.

Now let $\sigma' \in I(\sigma(Q))$. That is, $\sigma'(x) - x \in \sigma(Q)$ for all $x \in B$. In particular, $\sigma'\sigma(x) - \sigma(x) \in \sigma(Q)$, so $\sigma^{-1}\sigma'\sigma(x) - x \in Q$, so $\sigma^{-1}\sigma'\sigma \in I(Q)$, so $\sigma' \in \sigma I(Q)\sigma^{-1}$.

Thus $I(\sigma(Q)) = \sigma I(Q)\sigma^{-1}$. □

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If L/K is an abelian extension, show that the groups $D(\sigma(Q))$ are all equal, as are the $I(\sigma(Q))$. Show that the groups depend only on the prime ideal $P = Q \cap A$.

Proof. Under the assumption that the Galois group is abelian, we have, by the previous result, $D(\sigma(Q)) = \sigma D(Q)\sigma^{-1} = \sigma\sigma^{-1}D(Q) = D(Q)$, and similarly for $I(\sigma(Q))$. Since any two prime ideals Q_1, Q_2 which share $P = Q_1 \cap A = Q_2 \cap A$ must be Galois conjugates, $Q_1 = \sigma(Q_2)$, we then have $D(Q_1) = D(\sigma(Q_2)) = D(Q_2)$. Thus, the groups only depend on the base prime ideal. □