MATH 7230 Homework

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1 Problem 1

Let $K = \mathbb{Q}(\sqrt{m})$, where $m = d_K$. Write down the definition of the Dedekind zeta function $\zeta_K(s)$. Verify that $\zeta_K(s) = \zeta(s)L(\chi_m, s)$, where χ_m is the character defined by $\chi_m(p) = (\frac{m}{p}) = m^{(p-1)/2}$ for primes $p \nmid m$.

Proof. $\zeta_K(s) = \sum N(I)^{-s}$, where I ranges over all non-zero ideals of the ring of integers $\mathcal{O}_K$. Using prime factorization of ideals in the Dedekind domain $\mathcal{O}_K$ and the multiplicative property of the norm, we have $\zeta_K(s) = \prod(1-N(P)^{-s})^{-1}$, where P ranges over all the non-zero prime ideals of $\mathcal{O}_K$. The prime ideals are classified by factorization of the minimal polynomial of θ mod p, where $\mathcal{O}_K = \mathbb{Z}[\theta]$ and p is a prime in $\mathbb{Z}$. When $p \mid m$, $p\mathcal{O}_K = \mathcal{P}^2$, so there is one prime ideal of norm p. When $p \nmid m$, $p\mathcal{O}_K = \mathcal{P}_1\mathcal{P}_2$ or $p\mathcal{O}_K = \mathcal{P}$, depending on whether $(\frac{m}{p}) = 1$ or -1, respectively. In the splitting case, we have two prime ideals of norm p, and in the inert case, we have one prime ideal of norm p^2. Thus

$$\zeta_K(s) = \prod_{p|m}(1-p^{-s})^{-1} \prod_{(\frac{m}{p})=1}(1-p^{-s})^{-2} \prod_{(\frac{m}{p})=-1}(1-p^{-2s})^{-1}$$

$$= \prod_{p}(1-p^{-s})^{-1} \prod_{(\frac{m}{p})=1}(1-p^{-s})^{-1} \prod_{(\frac{m}{p})=-1}(1+p^{-s})^{-1}$$

$$= \zeta(s)\prod_{p}(1-\left(\frac{m}{p}\right)p^{-s})^{-1} = \zeta(s)L(\chi_m, s).$$

□

2 Problem 2

Let $m = -7$. Write down the first few terms of $L(\chi_m, s)$ as a series. Write a program to compute the numerical value of $L(\chi_m, 1)$ up to the first 4 digits after the decimal using the series expansion. Write down the precise formula for $L(\chi_m, 1)$ and check whether they agree up to the first 4 digits after the decimal.

Proof. $L(\chi_m, s) = \prod_{p\nmid m}(1-\chi_m(p)p^{-s})^{-1} = (1-2^{-s})^{-1}(1+3^{-s})^{-1}(1+5^{-s})^{-1}(1-11^{-s})^{-1}(1+13^{-s})^{-1}... = 1+2^{-s}-3^{-s}+4^{-s}-5^{-s}-6^{-s}+8^{-s}+9^{-s}-10^{-s}+11^{-s}-12^{-s}-13^{-s}+....$ Plugging in $s = 1$ for the series gives 1.1168. Plugging in $s = 1$ into the product gives 1.2769. The average of these two is 1.1969. Using the Wolfram code

$$N[1/Product[1 - JacobiSymbol[-7, Prime[n]]/Prime[n], n, 100000]]$$

gives 1.1875.

From the class number formula, we have $L(\chi_m, 1) = 2^{r_1}(2\pi)^{r_2}\dfrac{h_K R_K}{w_K|d_K|^{1/2}}$. First, $m = -7$, so there are no real embeddings. The field is quadratic, so $r_2 = 1$. $d_K = -7$. The Minkowski bound is $2\sqrt{7}/\pi < 2$, so $h_K = 1$. The ring of integers is $\mathbb{Z}[\frac{1+\sqrt{-7}}{2}]$. Since $r = r_1 + r_2 - 1 = 0 + 1 - 1 = 0$, there are finitely many units, which are the roots of unity. This also implies that $R_K = 1$. To find roots of unity, we employ the norm: $N(a + b\frac{1+\sqrt{-7}}{2}) = \pm 1$, which gives $a^2 + ab + 2b^2 = \pm 1$. The left hand side is also equal to $(a + b)^2 + b^2 - ab$, so that $a^2 + (a + b)^2 + 3b^2 = \pm 2$. The left hand side is obviously non-negative for $a, b \in \mathbb{Z}$, so the -2 case is null. Furthermore, if $b \neq 0$, then the left hand side is at least 3. Thus $b = 0$ which gives $a^2 = 1$, so there are only $w_K = 2$ roots of unity, ± 1. Thus $L(\chi_m, 1) = 2\pi/(2\sqrt{7}) = 1.1874$. This is a very good match with the analytic expansion up to 100000 terms, but not so much for the first couple of terms in the series or product. The average of the two is closer. $\square$

3 Problem 3

Let $m = 28$. Write down the first few terms of $L(\chi_m, s)$ as a series. Write a program to compute the numerical value of $L(\chi_m, 1)$ up to the first 4 digits after the decimal using the series expansion. Write down the precise formula for $L(\chi_m, 1)$ and check whether they agree up to the first 4 digits after the decimal.

Proof. Note first that $\mathbb{Q}(\sqrt{28}) = \mathbb{Q}(\sqrt{7})$. $L(\chi_m, s) = (1-3^{-s})^{-1}(1+5^{-s})^{-1}(1+11^{-s})^{-1}(1+13^{-s})^{-1}... = 1+3^{-s}-5^{-s}+9^{-s}-11^{-s}-13^{-s}-15^{-s}+....$ Plugging in $s = 1$ for the series gives 1.0099. Plugging in $s = 1$ for the product gives 1.0640. The average is 1.0370. Using the Wolfram code

$$N[1/Product[1 - JacobiSymbol[28, Prime[n]]/Prime[n], n, 100000]]$$

gives 1.0465.

Now, K is a real quadratic field, so $r_1 = 2, r_2 = 0$. $d_K = 28$. The Minkowski bound is $\sqrt{7} < 3$. We must then check the ideals of norm 2. The ring of integers is $\mathbb{Z}[\sqrt{7}]$. Thus $(2) = (2, \sqrt{7}+1)^2$, so $(2, \sqrt{7}+1)$ is the only ideal of norm 2. However, the element $3+\sqrt{7}$ has norm $3^2-7\cdot 1^2 = 2$, so $(3+\sqrt{7})$ is also an ideal of norm 2, and it is principal. Thus $h_K = 1$. Since the field is real, the only roots of unity are ± 1, so $w_K = 2$. Now, the units are a finitely generated module of rank $r = r_1+r_2-1 = 2+0-1 = 1$. The fundamental unit is $8+3\sqrt{7}$. The regulator is then $R_K = \ln(8+3\sqrt{7}) = 2.7687$. Thus $L(\chi_m, 1) = 2^2\frac{2.7687}{2\sqrt{28}} = 1.0465$, which is a perfect match with the analytic expansion up to 100000 terms, to 4 decimal places. Once again, the first few terms of the product or series do not match well, but the average is closer. $\square$