MATH 7230 Homework 7

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1 Problem 7.1 1

Show that the sign of $D(1, \alpha, ...\alpha^{n-1})$ is the sign of $\prod_{i=1}^{r_2}(c_{r_1+i} - \overline{c_{r_1+i}})^2$.

Proof. The discriminant is $\prod_{i<j}(\sigma_i(\alpha) - \sigma_j(\alpha))^2$. One part of this product is the terms $(c_i - c_j)^2$ where $i, j \leq r_1$. Since the c_i are real for $i \leq r_1$, $(c_i - c_j)^2 > 0$. Another part of this product is the terms $(c_i - c_j)^2$ where $i \leq r_1 < j$. Then there is the respective term $(c_i - \overline{c_j})^2$, which is exactly the conjugate of $(c_i - c_j)^2$ since c_i is real. Thus the two multiply to give a positive number. Similarly for $r_1 < i, j$, there are two pairs of conjugate terms: $(c_i - c_j)^2$, $(\overline{c_i} - \overline{c_j})^2$ and $(c_i - \overline{c_j})^2$, $(\overline{c_i} - c_j)^2$. The only terms which are left unpaired are those $(c_i - \overline{c_i})^2$, as desired. $\square$

2 Problem 7.1 2

Show that the sign of the discriminant is $(-1)^{r_2}$.

Proof. Write $c_{r_1+i} = a_i + ib_i$. Then $(c_{r_1+i} - \overline{c_{r_1+i}})^2 = (a_i + ib_i - a_i + ib_i)^2 = (2ib_i)^2 = -4b_i^2 < 0$. Thus there are r_2 negative terms, so the sign is $(-1)^{r_2}$. $\square$

3 Problem 7.1 3

Apply the results to $\alpha = \zeta$, where ζ is a primitive p^rth root of unity.

Proof. $r_2 = \frac{1}{2}n = \frac{1}{2}\varphi(p^r) = \frac{1}{2}p^{r-1}(p-1)$. p^{r-1} is odd, so $(-1)^{r_2} = -(-1)^{(p-1)/2} = (-1)^{(p+1)/2}$. □