MATH 7230 Homework 6

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For Problems 6.1 1-5, let H be a discrete subgroup of $\mathbb{R}^n$. For $r \leq n$ as large as possible, let $e_1, ..., e_r \in H$ be linearly independent over $\mathbb{R}$. Let T be the fundamental domain determined by the e_i. Since H is discrete, $H \cap \overline{T}$ is finite. Now let $x \in H$ with $x = \sum_{i=1}^r b_i e_i$.

1 Problem 6.1 1

If j is any integer, set $x_j = jx - \sum_{i=1}^r \lfloor jb_i \rfloor e_i$. Show that $x_j \in H \cap \overline{T}$.

Proof. $x_j = \sum_{i=1}^r (jb_i - \lfloor jb_i \rfloor) e_i = \sum_{i=1}^r \{jb_i\} e_i$, where $\{x\}$ represents the fractional part of x, which is a real number in the interval $[0, 1)$. Thus $x_j \in \overline{T}$. Furthermore, x_j is a linear combination of elements in H, so it is also an element of H. $\square$

2 Problem 6.1 2

By examining the above formula with $j = 1$, show that H is a finitely generated $\mathbb{Z}-$module.

Proof. $x_1 = \sum_{i=1}^{r} \{b_i\} e_i \in H \cap \overline{T}$, which is a finite set. Thus, every element of H can be expressed as $f_k + \sum_{i=1}^{r} \lfloor b_i \rfloor e_i$, where $f_k \in H \cap \overline{T}$. Since the $e_i \in \overline{T}$ as well, it follows that $H \cap \overline{T}$ is a finite generating set for H over $\mathbb{Z}$. $\square$

3 Problem 6.1 3

Show that the b_i are rational numbers.

Proof. The element x gives an infinite sequence of elements x_i in the finite set $H \cap \overline{T}$. Therefore, there are some positive integers j and k such that $x_j = x_k$. Then $jx - \sum_{i=1}^{r} \lfloor jb_i \rfloor e_i = kx - \sum_{i=1}^{r} \lfloor kb_i \rfloor e_i$, so $(j-k)x = \sum_{i=1}^{r} (\lfloor jb_i \rfloor - \lfloor kb_i \rfloor) e_i$. By linear independence of the e_i, $b_i = \dfrac{\lfloor jb_i \rfloor - \lfloor kb_i \rfloor}{j-k} \in \mathbb{Q}$. □

4 Problem 6.1 4

Show that for some nonzero integer d, dH is a free $\mathbb{Z}-$module of rank at most r.

Proof. Since every element in H has coordinates in $\mathbb{Q}$ over the e_i, it follows that there are finitely many rational numbers comprising the coordinates of the elements in $H \cap \overline{T}$. Thus we can take a common denominator d of the rational numbers, so that dH consists only of $\mathbb{Z}$ linear combinations of the e_i, since any element of H has coordinates with fractional parts equal to the rational numbers comprising $H \cap \overline{T}$. $\square$

5 Problem 6.1 5

Show that H is a lattice in $\mathbb{R}^r$.

Proof. $\square$

6 Problem 6.3 1

Calculate the fundamental unit of $\mathbb{Q}(\sqrt{m})$ for $m = 2, 3, 5, 6, 7, 10, 11, 13, 14, 15, 17$.

Proof. $m = 2$: $2 \cdot 1^2$ is 1 away from 1^2, so $1 + \sqrt{2}$ is the fundamental unit.

$m = 3$: $3 \cdot 1^2$ is 1 away from 2^2, so $2 + \sqrt{3}$ is the fundamental unit.

$m = 5$: $5 \cdot 1^2$ is 4 away from 1^2, so $\frac{1}{2}(1 + \sqrt{5})$ is the fundamental unit.

$m = 6$: $6 \cdot 2^2$ is 1 away from 5^2, so $5 + 2\sqrt{6}$ is the fundamental unit.

$m = 7$: $7 \cdot 3^2$ is 1 away from 8^2, so $8 + 3\sqrt{7}$ is the fundamental unit.

$m = 10$: $10 \cdot 1^2$ is 1 away from 3^2, so $3 + \sqrt{10}$ is the fundamental unit.

$m = 11$: $11 \cdot 3^2$ is 1 away from 10^2, so $10 + 3\sqrt{10}$ is the fundamental unit.

$m = 13$: $13 \cdot 3^2$ is 4 away from 11^2, so $\frac{1}{2}(11 + 3\sqrt{13})$ is the fundamental unit.

$m = 14$: $14 \cdot 4^2$ is 1 away from 15^2, so $15 + 4\sqrt{14}$ is the fundamental unit.

$m = 15$: $15 \cdot 1^2$ is 1 away from 4^2, so $4 + \sqrt{15}$ is the fundamental unit.

$m = 17$: $17 \cdot 2^2$ is 4 away from 8^2, so $\frac{1}{2}(8 + 2\sqrt{17}) = 4 + \sqrt{7}$ is the fundamental unit. $\square$