MATH 7230 Homework 5

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1 Problem 5.3 1

Calculate the Minkowski bound on ideal norms for an imaginary quadratic field, in terms of the field discriminant d. Show that $\mathbb{Q}(\sqrt{m})$ has class number 1 for $m = -1, -3, -7$.

Proof. Since the extension is quadratic, $n = 2$. Since it is imaginary, $r_1 = 0$, so $r_2 = 1$. Thus the Minkowski bound is $(4/\pi)(2!/2^2)|d|^{1/2} = \dfrac{2|d|^{1/2}}{\pi}$.

For $m = -1$, $d = -4$, so the bound is $4/\pi < 2$. Thus any ideal class contains the whole ring B, so there is one ideal class.

For $m = -3$, $d = -3$, so the bound is $2\sqrt{3}/\pi < 2$, so again there is only one ideal class.

For $m = -7$, $d = -7$, so the bound is $2\sqrt{7}/\pi < 2$, so again there is only one ideal class. □

2 Problem 5.3 2

Calculate the Minkowski bound on ideal norms for a real quadratic field, in terms of the field discriminant d. Use the result to show that $\mathbb{Q}(\sqrt{m})$ has class number 1 for $m = 2, 3, 5, 13$.

Proof. For a real quadratic field, $n = 2, r_1 = 2, r_2 = 0$. Thus the bound is $(2!/2^2)|d|^{1/2} = |d|^{1/2}/2$.

For $m = 2$, $d = 8$, so the bound is $\sqrt{2} < 2$. Once again, there is one ideal class.

For $m = 3$, $d = 12$, so the bound is $\sqrt{3} < 2$. Once again, there is one ideal class.

For $m = 5$, $d = 5$, so the bound is $\sqrt{5}/2 < 2$, so there is just one ideal class again.

For $m = 13$, $d = 13$, so the bound is $\sqrt{13}/2 < 2$, so there is one ideal class again. □

3 Problem 5.3 3

Show that in the ring of algebraic integers of $\mathbb{Q}(\sqrt{-5})$, there is only one ideal whose norm is 2. Then use the Minkowski bound to prove that class number is 2.

Proof. Recall that the ring of algebraic integers in this case is $\mathbb{Z}[\sqrt{-5}]$. The element 2 has norm $2^2 = 4$. The prime factorization of the ideal (2) is $(2, 1 + \sqrt{-5})^2$, since $X^2 - 5 \equiv (X+1)^2 \bmod 2$. Thus the ideal $(2, 1+\sqrt{-5})$ is the only ideal with norm 2; a norm 2 ideal must divide (2), but the only divisors are $(1), (2, 1 + \sqrt{-5})$, and (2). The norms of (1) and (2) are 1 and 4, respectively. This leaves only $(2, 1 + \sqrt{-5})$.

Now, the field discriminant is $d = -20$, so the Minkowski bound is $(2/\pi)\sqrt{20} \approx 2.847 < 3$. Thus any ideal class contains an integral ideal with norm 1 or 2. If the norm is 1, we know that the class will be trivial, since norm 1 implies the ideal is equal to the entire ring. With norm 2, we know that the ideal will be $(2, 1 + \sqrt{-5})$.

We must verify that this is not principal. We can look at the norms of 2 and $1 + \sqrt{-5}$: the norms are 4 and 6, respectively. But $a^2 + 5b^2 \neq 2$, since for $b \neq 0$ the quantity is at least 5, and for $b = 2$, $a^2 = 2$ is not solvable in integers. Thus the ideal is not principal, and so it cannot be in the same ideal class group as the entire ring of integers. Thus, the class number is 2. □

4 Problem 5.3 6

Find the class number of $\mathbb{Q}(\sqrt{14})$.

Proof. The discriminant is 56. The Minkowski bound is then $\sqrt{14} \approx 3.74$. We must look at ideals of norm 2 and 3. The ring of integers is $\mathbb{Z}[\sqrt{14}]$.

An ideal of norm 2 will divide (2). The factorization is $(2) = (2, \sqrt{14})^2$ since $X^2 - 14 \equiv X^2 \bmod 2$. Thus $(2, \sqrt{14})$ is the only ideal of norm 2. However, $4 - \sqrt{14}$ is an element of norm $4^2 - 14 = 2$, and $2 = (4 + \sqrt{14})(4 - \sqrt{14})$ and $\sqrt{14} = (7+2\sqrt{14})(4-\sqrt{14})$. Thus $(2, \sqrt{14}) = (4-\sqrt{14})$ is principal, and belongs to the trivial ideal class.

An ideal of norm 3 will divide (3). $X^2 - 14 \equiv X^2 + 1 \bmod 3$ is irreducible, since a square mod 3 is either 0 or 1. Thus $(3) = (3, 15) = (3)$ is prime. But the norm of (3) is 9, so there are no ideals of norm 3.

Thus the class number is 1. $\square$