MATH 7230 Homework 4

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1 Problem 4.2 1

Let $(p) = pB$ have prime factorization $\Pi_i P_i^{e_i}$. Show that p ramifies if and only if the ring $B/(p)$ has nonzero nilpotent elements.

Proof. If p does not ramify, then B/pB is isomorphic to a direct sum of finite fields, and so it has no nonzero nilpotent elements. Conversely, if p ramifies, then it is a direct sum of finite rings, at least one not being an integral domain, say $B/P_i^{e_i}$. Then the element $p_i + P_i^{e_i}$ for $p_i \in P_i \setminus P_i^{e_i}$ is nonzero and nilpotent; $(p_i + P_i^{e_i})^{e_i} = p_i^{e_i} + P_i^{e_i} = 0 + P_i^{e_i}$. Note also that $P_i \setminus P_i^{e_i}$ is not empty, because then $P_i^{e_i}$ would be prime, a contradiction. This extends to an element of B/pB which is nilpotent and non-zero, say $(0 + P_1^{e_1}, ..., p_i + P_i^{e_i}, ..., 0 + P_r^{e_r})$. $\square$

2 Problem 4.2 2

Show that a nilpotent element (or matrix) has zero trace.

Proof. A nilpotent element satisfies a polynomial equation $X^n = 0$. The minimal polynomial must then divide X^n, so it is equal to X^d. The characteristic polynomial is a power of the minimal polynomial, so it is equal to some X^k. Thus the trace is zero. $\square$

3 Problem 4.2 3

Represent elements of B by matrices in terms of an integral basis $\omega_1, ..., \omega_n$. Reduction of the entries mod p gives matrices representing elements of $B/(p)$. Suppose $A(\beta)$, the matrix representing β, is nilpotent mod p. Then $A(\beta\omega_i)$ is nilpotent mod p for each i since $\beta\omega_i$ is nilpotent mod p. By expressing β in terms of the ω_i and computing the trace of $A(\beta\omega_j)$, show that if β is nilpotent mod p and $\beta \notin (p)$, then $d \equiv 0 \bmod p$.

Proof. Recall that, by definition, $Tr(A(\beta)) = Tr(\beta)$. Let $\beta = \sum b_i\omega_i$. If β is nilpotent mod p, then by the above problem, $Tr(\beta) \equiv 0 \bmod p$. Furthermore, if $\beta^n \equiv 0 \bmod p$, then $(\beta\omega_j)^n = \beta^n\omega_j^n \equiv 0 \bmod p$. Thus $\beta\omega_j$ is also nilpotent and has a trace that is 0 mod p. Specifically, $Tr(\beta\omega_j) = Tr(\sum b_i\omega_i\omega_j) = \sum b_i Tr(\omega_i\omega_j) \equiv 0 \bmod p$. This holds for each $j = 1, ..., n$. Since $\beta \notin (p)$, not all the b_i are 0 mod p. Thus by manipulating the rows of the matrix $(Tr(\omega_i\omega_j))$, we form a matrix with a row of elements which are 0 mod p, so it has a determinant which is 0 mod p. Since we can do this with b_i which are not 0 mod p, it follows that the determinant of $(Tr(\omega_i\omega_j))$ is also 0 mod p. By definition, $d = \det(Tr(\omega_i\omega_j))$, so $d \equiv 0 \bmod p$. □

4 Problem 4.3 1

Using the results of the section, factor (2) and (3) in the ring $\mathbb{Z}[\sqrt{-5}]$ rigorously.

Proof. The minimal polynomial of $\sqrt{-5}$ is $X^2 + 5$. Mod 2 gives $X^2 + 1$ which factors into $(X + 1)^2$. Thus $(2) = (2, 1 + \sqrt{-5})^2$.

Mod 3 gives $X^2 + 2$ which factors into $(X + 1)(X + 2)$. Thus $(3) = (2, 1 + \sqrt{-5})(2, 2 + \sqrt{-5})$. □

5 Problem 4.3 2

Factor $(5), (7)$, and (11) in $\mathbb{Z}[\sqrt{-5}]$.

Proof. Mod 5 gives X^2. Thus $(5) = (5, \sqrt{-5})^2 = (\sqrt{-5})^2$.

Mod 7 gives $X^2 + 5 = (X + 3)(X + 4)$. Thus $(7) = (5, 3 + \sqrt{-5})(5, 4 + \sqrt{-5})$.

Mod 11 gives $X^2 + 5$, which cannot be factored because 6 is not a quadratic residue mod 11. Thus (11) is a prime ideal in $\mathbb{Z}[\sqrt{-5}]$. □

6 Problem 4.3 3

Let $L = \mathbb{Q}(\sqrt[3]{2})$, and assume that the ring of integers is $\mathbb{Z}[\sqrt[3]{2}]$. Find the prime factorization of (5).

Proof. The minimal polynomial is $X^3 - 2$. Reducing mod 5 gives $X^3 + 3 = (X+2)(X^2+3X+4)$. Thus $(5) = (5, 2+\sqrt[3]{2})(5, 4+3\sqrt[3]{2}+\sqrt[3]{4})$. □