MATH 7230 Homework 3

Andrea Bourque

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For Problems 3.1 1-3: Let $P_1, P_2, ..., P_s$, $s \geq 2$ be ideals in a ring R, with P_1, P_2 not necessarily prime, but $P_3, ..., P_s$ prime, if $s \geq 3$. Now let I be any ideal of R. We claim that if $I \subseteq \cup_{i=1}^{s} P_i$, then for some i we have $I \subseteq P_i$.

1 Problem 3.1 1

Suppose the result is false. Show that without loss of generality, we can assume the existence of elements $a_i \in I$ with $a_i \in P_i$ but $a_i \notin P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_s$.

Proof. Each element of I is contained in some P_i, by the property of unions. If there is some P_i for which $I \cap P_i = \emptyset$, then we may remove P_i from our collection and relabel the ideals. This is valid because I could never be a subset of P_i, and the fact that I is a subset of the union would not change. Thus, without loss of generality, for each i, there is some $a_i \in I$ with $a_i \in P_i$.

Now, suppose it is not possible to choose $a_i \notin P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_s$. Then the ideal $I \cap P_i \subseteq P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_s$. This is the hypothesis of the lemma, with $I \cap P_i$ taking the place of I, but with one less ideal. We then could apply mathematical induction, assuming we prove a base case, which will be done in the next problem. Thus, we know that $I \cap P_i \subseteq P_j$ for some $j \neq i$. But then $I \subseteq P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_s$, since P_j contains $I \cap P_i$ and I is contained in $(P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_s) \cup P_i$. Therefore, if we cannot choose $a_i \in I$ with $a_i \in P_i$ and $a_i \notin P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_s$, we can discard P_i altogether. Thus, without loss of generality, we can assume the existence of elements $a_i \in I$ with $a_i \in P_i$ and $a_i \notin P_1 \cup \cdots \cup P_{i-1} \cup P_{i+1} \cup \cdots \cup P_s$. □

2 Problem 3.1 2

Prove the result for $s = 2$.

Proof. Let $I \subseteq P_1 \cup P_2$, and suppose $I \not\subseteq P_1$, $I \not\subseteq P_2$. If $x \in I$, then either $x \in P_1$ or $x \in P_2$. But there must be some $x_1, x_2 \in I$ with $x_1 \notin P_2$, $x_2 \notin P_1$, so that $x_1 \in P_1$, $x_2 \in P_2$. Then $x_1 + x_2 \in I$, so either $x_1 + x_2 \in P_1$ or $x_1 + x_2 \in P_2$. But this is equivalent to $x_2 \in P_1$ or $x_1 \in P_2$, which is a contradiction. □

3 Problem 3.1 3

Now assume $s > 2$, and observe that $a_1 a_2 \dots a_{s-1} \in P_1 \cap \dots \cap P_{s-1}$ but $a_s \notin P_1 \cup \dots \cup P_{s-1}$. Let $a = a_1 a_2 \dots a_{s-1} + a_s$, which does not belong to $P_1 \cup \dots \cup P_{s-1}$. Show that $a \in I$ and $a \notin P_1 \cup \dots \cup P_s$, a contradiction.

Proof. Each $a_i \in I$, so by properties of ideals we have $a \in I$ as well. We know $a \notin P_1 \cup \dots \cup P_{s-1}$, since $a_1 a_2 \dots a_{s-1} \in P_1 \cup \dots \cup P_{s_1}$ and $a_s \notin P_1 \cup \dots \cup P_{s-1}$. Then if $a \in P_1 \cup \dots \cup P_s$, the only possibility would be $a \in P_s$. Since $a_s \in P_s$, this implies $a_1 \dots a_{s-1} \in P_s$. Since $s > 2$, P_s is a prime ideal. This means that some $a_i \in P_s$, where $i = 1, ..., s-1$. This is a contradiction, since each $a_i \notin P_1 \cup \dots \cup P_{i-1} \cup P_{i+1} \cup \dots \cup P_s$. □

4 Problem 3.2 1

If I and J are relatively prime ideals, show that $IJ = I \cap J$. More generally, if $I_1, ..., I_n$ are relatively prime in pairs, show that $I_1 \dots I_n = \cap_{i=1}^n I_i$.

Proof. In general, we know that $IJ \subseteq I \cap J$ by properties of ideals. Thus let $x \in I \cap J$. Suppose $1 = r + s$, where $r \in I, s \in J$. Then $x = rx + xs$ shows that $x \in IJ$.

In the general case, we use induction. We must show that if I_n is coprime to each of $I_1, ...I_{n-1}$, then I_n is coprime to $I_1 \cap \dots \cap I_{n-1}$. We can create a set of equations $a_i + b_i = 1$, where $a_i \in I_i$ and $b_i \in I_n$, for $i = 1, ..., n-1$. By expanding $\Pi_{i=1}^n (a_i + b_i) = 1$, we get $a + b = 1$, where $a = a_1 \dots a_n$ and b are all the other terms. It is clear that $a \in I_1 \cap \dots \cap I_{n-1}$ and $b \in I_n$ by the definition of ideals. Thus I_n and $I_1 \cap \dots \cap I_{n-1}$ are coprime, so we can inductively prove the statement since $I_1 \dots I_n = (I_1 \cap \dots \cap I_{n-1}) I_n = (I_1 \cap \dots \cap I_{n-1}) \cap I_n = I_1 \cap \dots \cap I_n$. □

5 Problem 3.2 2

Let P_1 and P_2 be relatively prime ideals. Show that P_1^r and P_2^s are relatively prime for arbitrary positive integers r, s.

Proof. If $a + b = 1$, where $a \in P_1, b \in P_2$, then

$$1 = (a+b)^{r+s-1} = a^r \left(a^{s-1} + ... + \binom{r+s-1}{r} b^{s-1} \right) + b^s \left(\binom{r+s-1}{r+1} a^{r-1} + ... + b^{r-1} \right)$$

shows that $1 \in P_1^r + P_2^s$, so P_1^r and P_2^s are coprime. □

6 Problem 3.3 4

Show that the ring of algebraic integers in $\mathbb{Q}(\sqrt{-17})$ is not a unique factorization domain.

Proof. The ring of algebraic integers is $\mathbb{Z}[\sqrt{-17}]$. Then $(2+\sqrt{-17})(2-\sqrt{-17}) = 3 \times 7$.

First, we show that no elements have $N(x) = 3$ or 7. For, suppose $x = a + b\sqrt{-17} \in \mathbb{Z}[\sqrt{-17}]$, so $N(x) = a^2 + 17b^2$. If $b \neq 0$, then $N(x) \geq 17$. Then $N(x) = 3$ or 7 implies $b = 0$, so $x \in \mathbb{Z}$. But, no square of an integer is equal to 3 or 7, so the norm of an element cannot possibly be 3 or 7.

Next, we show that if $N(x) = 1$, then $x = \pm 1$. Using the same inequality that $N(x) \geq 17$ if $b \neq 0$, we must have $b = 0$, so that $x \in \mathbb{Z}$. Then x is a square root of 1, so it is just ± 1.

Now, $N(2 + \sqrt{-17}) = 21$. If $2 + \sqrt{-17} = ab$, then $N(a)N(b) = 21$. We have ruled out the possibility for the norms to be 3 or 7, so without loss of generality, we have $N(a) = 1$ and $N(b) = 21$. But then $a = \pm 1$, and $b = \pm(2 + \sqrt{-17})$. Therefore, $2 + \sqrt{-17}$ is irreducible. Similar logic shows that $2 - \sqrt{-17}$ is irreducible, since it also has a norm of 21.

Next, $N(3) = 9$, so if $3 = ab$, $N(a)N(b) = 9$. Again, $N(a) \neq 3$, so without loss of generality, $N(a) = 1$, and thus $a = \pm 1$, $b = \pm 3$.

Finally, $N(7) = 49$, so if $7 = ab$, $N(a)N(b) = 49$. $N(a) \neq 7$, so without loss of generality, $N(a) = 1$, $a = \pm 1$, $b = \pm 7$.

Therefore, in the ring $\mathbb{Z}[\sqrt{-17}]$, 21 has two unique factorizations, so the ring is not a unique factorization domain. □

For the problems 3.4 1-3, let P_2 be the ideal $(2, 1+\sqrt{-5})$ in $\mathbb{Z}[\sqrt{-5}]$.

7 Problem 3.4 1

Show that $1-\sqrt{-5} \in P_2$ and conclude that $6 \in P_2^2$.

Proof. $1-\sqrt{-5} = 2-(1+\sqrt{-5}) \in P_2$. Then $6 = (1+\sqrt{-5})(1-\sqrt{-5}) \in P_2^2$. □

8 Problem 3.4 2

Show that $2 \in P_2^2$, and hence that $(2) \subseteq P_2^2$.

Proof. We have $6 \in P_2^2$ by the above result. Furthermore, since $2 \in P_2$, $4 = 2^2 \in P_2^2$. Thus $2 = 6 - 4 \in P_2^2$ as desired. □

9 Problem 3.4 3

Expand $P_2^2 = (2, 1 + \sqrt{-5})(2, 1 + \sqrt{-5})$ and conclude that $P_2^2 \subseteq (2)$.

Proof. We have

$$\begin{aligned}(2a+b(1+\sqrt{-5}))(2c+d(1+\sqrt{-5}) &= 4ac+2(ad+bc)(1+\sqrt{-5})+bd(-4+2\sqrt{-5}) \\ &= 2\left[2ac + ad + bc - 2bd + (ad + bc + bd)\sqrt{-5}\right] \in (2),\end{aligned}$$

so any element of P_2^2 is a sum of multiples of 2, so $P_2^2 \subseteq (2)$. □