MATH 7230 Homework 10

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1 Problem 9.1 1

Show that every absolute value on a finite field is trivial.

Proof. Assume that there is a non-trivial absolute value $|\cdot|$ on a finite field k. Then there exists some $x \in k$ such that $|x| \neq 0$ and $|x| \neq 1$. Then either $|x| < 1$ or $|x| > 1$. If $|x| < 1$, then since $|x||x^{-1}| = |1| = 1$, we have $|x^{-1}| > 1$. In either case, we have an element $y \in k$ such that $|y| > 1$. Then the sequence $|y|, |y|^2, |y|^3, ...$ is a strictly monotone sequence in $\mathbb{R}$, taking on infinitely many distinct values. However, this is the same as the sequence $|y|, |y^2|, |y^3|, ...$ which consists of absolute values of elements in k; it is a sequence in $|k| = \{r \in \mathbb{R} \mid r = |x|, x \in k\}$. There are only finitely many elements in k, so there are only finitely many elements in $|k|$. Thus we have a contradiction. □

2 Problem 9.1 2

Show that a field that has an archimedean absolute value must have characteristic 0.

Proof. Suppose that a field of characteristic p has an archimedean absolute value $|\cdot|$. Then there is an integer n with $|n| > 1$. Then $|n|^p > |n|$. But $n^p = n$ in characteristic p, so $|n|^p = |n^p| = |n|$, a contradiction. □

3 Problem 9.1 3

Two nontrivial absolute values $|\cdot|_1$ and $|\cdot|_2$ on the same field are said to be equivalent if for every x, $|x|_1 < 1$ if and only if $|x|_2 < 1$. Show that two nontrivial absolute values are equivalent if and only if for some real number a, we have $|x|_1^a = |x|_2$ for all x.

Proof. First assume the absolute values are equivalent. Since the absolute values are non-trivial, let $y \in k$ with $|y|_1 > 1$. Thus $|y|_2 > 1$. Let $a = \log|y|_2/\log|y|_1$, so $|y|_1^a = |y|_2$. Now let $x \in k$ and $b = \log|x|_1/\log|y|_1$, so $|x|_1 = |y|_1^b$. Let $\varepsilon > 0$. Then $|x|_1 < |y|_1^{b+\varepsilon}$, so $|x/y^{b+\varepsilon}|_1 < 1$. Thus $|x/y^{b+\varepsilon}|_2 < 1$. Letting $\varepsilon \to 0$, we get $|x/y^b|_2 \leq 1$, or $|x|_2 \leq |y|_2^b$. Similarly, $|y|_1^{b-\varepsilon} < |x|_1$ implies $|y|_2^{b-\varepsilon} < |x|_2$, and taking $\varepsilon \to 0$ gives $|x|_2 \geq |y|_2^b$. Therefore, $|x|_2 = |y|_2^b$. Thus $b = \log|x|_2/\log|y|_2 = \log|x|_1/\log|y|_1$, so $a = \log|y|_2/\log|y|_1 = \log|x|_2/\log|x|_1$, so $|x|_1^a = |x|_2$. □

4 Problem 9.2 1

If a is a non-zero rational number, then $\prod_p ||a||_p = 1$, where p ranges over all primes, including $p = \infty$, corresponding to the usual absolute value.

Proof. By the fundamental theorem of arithmetic, $a = \pm p_1^{e_1} \dots p_k^{e_k}$. For a prime $p|a$, $p = p_i$ for some $i = 1, ..., k$. Then $||a||_p = p_i^{-e_i}$. If a prime $p \nmid a$, then $||a||_p = 1$. Thus $\prod_p ||a||_p = |a| \prod_{p|a} ||a||_{p_i} = p_1^{e_1} \dots p_k^{e_k} \prod_{i=1}^k p_i^{-e_i} = 1$. $\square$

5 Problem 9.3 1

1. Let $|\cdot|_1, ..., |\cdot|_n$ be nontrivial mutually inequivalent absolute values on the field k. Fix r with $0 \leq r \leq n$. Show that there is an element $a \in k$ such that $|a|_1 > 1, ..., |a|_r > 1$ and $|a|_{r+1}, ..., |a|_n < 1$.

Proof. For each $i = 1, ..., n$, let $|y_i|_i > 1$ and $|z_i|_i < 1$. The existence of the elements $y_1, ..., y_n, z_{r+1}, ..., z_n$ is given by non-triviality of the absolute values. Now, for $i \leq r$, let $x_i = y_i$, and for $i > r$, let $x_i = z_i$. Let $\varepsilon > 0$. By the Artin-Whaples approximation theorem, there is $a \in k$ such that $|a - x_i|_i < \varepsilon$ for each i. If $i \leq r$, this means $|y_i|_i \leq |a - y_i|_i + |a|_i < \varepsilon + |a|_i$. If $i > r$, this means $|z_i|_i \geq -|a - z_i|_i + |a|_i > -\varepsilon + |a|_i$. Letting $\varepsilon < \min(|y_1|_1 - 1, ..., |y_r|_r - 1, 1 - |z_{r+1}|_{r+1}, ..., 1 - |z_n|_n)$ suffices; $|a|_i > |y_i|_i - \varepsilon > |y_i|_i - (|y_i|_i - 1) = 1$ for $i \leq r$, and $|a|_i < |z_i|_i + \varepsilon < |z_i|_i + (1 - |z_i|_i) = 1$ for $i > r$. □

6 Problem 9.3 2

There is a gap in the first paragraph of the proof of (9.3.2), which can be repaired by showing that the implication $|a|_1 < 1 \to |a|_2 < 1$ is sufficient for equivalence. Prove this.

Proof. □