MATH 7230 Homework 1

Andrea Bourque

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For the problems 1.1 1-3, let A be a subring of the integral domain B, with B integral over A.

1 Problem 1.1 1

Suppose B is a field, and let $a \in A$ be non-zero. Then $a^{-1} \in B$, so there is an equation

$$(a^{-1})^n + c_{n-1}(a^{-1})^{n-1} + ... + c_1 a^{-1} + c_0 = 0$$

with all the $c_i \in A$. Prove that $a^{-1} \in A$, so that A is a field.

Proof. Multiplying both sides by a^n gives

$$1 + c_{n-1}a + ... + c_1 a^{n-1} + c_0 a^n = 0.$$

Rearranging gives

$$1 = -c_{n-1}a - ... - c_0 a^n = a(-c_{n-1} - ... - c_0 a^{n-1}),$$

so

$$a^{-1} = -c_{n-1} - ... - c_0 a^{n-1},$$

which is an algebraic expression in A. Thus $a^{-1} \in A$. □

2 Problem 1.1 2

Now assume A is a field, and let $b \in B$ be non-zero. Then $A[b]$ is a finite-dimensional vector space over A. Let f be the A-linear transformation $f(z) = bz$ for $z \in A[b]$. Show that f is injective.

Proof. Suppose $f(z) = bz = 0$. $z \in A[b] \subseteq B$, and B is an integral domain, so either $b = 0$ or $z = 0$. b was chosen to be non-zero, so $z = 0$. Thus $\ker(f) = 0$, so f is injective. $\square$

3 Problem 1.1 3

Show that f as defined before is surjective as well, and conclude B is a field.

Proof. Since B is integral over A, let $b^n + a_{n-1}b^{n-1} + ... + a_1 b + a_0 = 0$, with all the $a_i \in A$. Then $\{1, b, ..., b^{n-1}\}$ is a basis for $A[b]$. Let $c_{n-1}b^{n-1} + ... + c_0 \in A[b]$. Since A is a field, a_0^{-1} exists in A. Let

$$z = -c_0 a_0^{-1} b^{n-1} + (c_{n-1} - c_0 a_{n-1} a_0^{-1}) b^{n-2} + ... + (c_1 - c_0 a_1 a_0^{-1}).$$

Then

$$bz = -c_0 a_0^{-1} b^n + (c_{n-1} - c_0 a_{n-1} a_0^{-1}) b^{n-1} ... + (c_1 - c_0 a_1 a_0^{-1}) b$$
$$= -c_0 a_0^{-1}(-a_{n-1}b^{n-1} - ... - a_1 b - a_0) + ... + (c_1 - c_0 a_1 a_0^{-1}) b$$
$$= c_{n-1} b^{n-1} + ... + c_1 b + c_0.$$

Thus f is surjective. However, $1 \in A[b]$, so that there is a unique $z \in A[b]$ such that $f(z) = bz = 1$, so that $z = b^{-1}$, so B is a field. □

4 Problem 1.2 1

Let $\mathcal{M}$ be a maximal ideal of R, and assume that for every $x \in \mathcal{M}$, $1 + x$ is a unit. Prove that R is a local ring.

Proof. Let $x \in R \setminus \mathcal{M}$. The ideal generated by x and $\mathcal{M}$ contains $\mathcal{M}$, but must be larger than $\mathcal{M}$ since $x \notin \mathcal{M}$. Since $\mathcal{M}$ is maximal, this means x and $\mathcal{M}$ generate R. Thus $1 = ax + m$ for some $m \in \mathcal{M}$ and some $a \in R$. Then $ax = 1 - m$ is a unit by hypothesis. Then x is also a unit, since there is some $b \in R$ such that $abx = 1$. Thus, $\mathcal{M}$ consists of all the non-units in R. Any proper ideal does not contain units, and is therefore contained in $\mathcal{M}$, so $\mathcal{M}$ is the unique maximal ideal in R. Thus R is a local ring. $\square$

5 Problem 1.2 2

Show that if p is prime and n is a positive integer, then $\mathbb{Z}/p^n\mathbb{Z}$ is a local ring with maximal ideal (p).

Proof. In $\mathbb{Z}/p^n\mathbb{Z}$, the units are elements coprime to p^n. Then the non-units are multiples of p, so (p) is the set of non-units in $\mathbb{Z}/p^n\mathbb{Z}$. Therefore, it is the unique maximal ideal, since every proper ideal in a ring must be contained in the set of non-units. □

6 Problem 1.2 3

For any field k, let R be the ring of rational functions f/g with $f, g \in k[X_1, ..., X_n]$ and $g(a) \neq 0$ where a is a fixed point in k^n. Show that R is a local ring and identify the maximal ideal.

Proof. Note that for $f/g \in R$ to be a unit, the one condition that must be satisfied is that the inverse g/f has $f(a) \neq 0$. Therefore, the set of non-units consists of f/g where $f(a) = 0$. This is in fact an ideal, since if f_1/g_1 and f_2/g_2 are two functions which vanish at a, then $f_1(a)g_2(a) + f_2(a)g_1(a) = 0 + 0 = 0$, so the set is additively closed, and further, if $f/g \in R$, then $f(a)f_1(a) = 0$, so $ff_1/(gg_1)$ is also in this set. Since the set of non-units is an ideal, it is maximal, since any proper ideal is contained in the set of non-units. This also shows that this maximal ideal is unique, since any other maximal ideal would be contained in it. Thus, R is a local ring with the maximal ideal consisting of rational functions vanishing at $a \in k^n$. □