# MATH 7220 Homework 8

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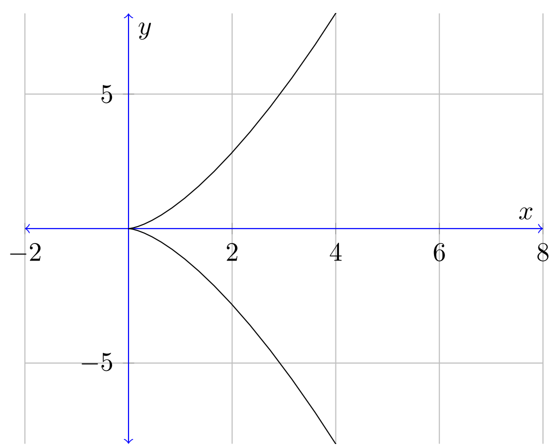
## 1 Problem 15.3

Show that $A = \mathbb{C}[x, y]/(y^2 - x^3)$ is an integral domain. Show that its field of fractions is $K = \mathbb{C}(y/x)$. Show that the integral closure of $A$ in $K$ is $\mathbb{C}[y/x]$. Draw the picture $y^2 - x^3 = 0$.

*Proof.* $y^2 - x^3$ is irreducible over $\mathbb{C}[x, y]$, so $(y^2 - x^3)$ is a prime ideal; thus, $A$ is an integral domain. Note that $x = x^3/x^2 = y^2/x^2 = (y/x)^2$, and $y = x(y/x) = (y/x)^3$. Thus any rational function in $x, y$ can be written as a rational function in $y/x$. Furthermore, this is obtained just by adjoining the inverse of $x$ into the ring, so $\mathbb{C}(y/x)$ is the smallest field in which elements of $A$ are invertible.

Write $t = y/x$. Then $A = \mathbb{C}[t^2, t^3]$ and $K = \mathbb{C}(t)$. $R := \mathbb{C}[y/x] = \mathbb{C}[t]$ is a UFD, so it is integrally closed in $K$, which is its field of fractions. $t$ is a root of the monic polynomial $X^2 - t^2$ in $A[X]$, i.e. $t$ is integral over $A$. Thus $R$ is integral over $A$. Since $R$ is integrally closed, $R$ is the integral closure of $A$ in $K$. $\square$



## 2 Problem 15.4

Show that the ring $A = \mathbb{C}[x, y]/(y^2 - x^2(x + 1))$ is an integral domain. Show that its field of fractions is $K = \mathbb{C}(y/x)$. Show that the integral closure of $A$ in $K$ is $\mathbb{C}[y/x]$. Draw the picture $y^2 - x^2(x + 1) = 0$.

*Proof.* $y^2-x^2(x+1)$ is irreducible over $\mathbb{C}[x, y]$, so $(y^2-x^2(x+1))$ is a prime ideal; thus, $A$ is an integral domain. Note that $x = x^3/x^2 = (y^2-x^2)/x^2 = (y/x)^2-1$, and $y = x(y/x) = (y/x)^3-y/x$. Thus any rational function in $x, y$ can be written as a rational function in $y/x$. Furthermore, this is obtained just by adjoining the inverse of $x$ into the ring, so $\mathbb{C}(y/x)$ is the smallest field in which elements of $A$ are invertible.

Write $t = y/x$. Then $K = \mathbb{C}(t)$ and $A = \mathbb{C}[t^2 - 1, t^3 - t]$. Let $R = \mathbb{C}[y/x] = \mathbb{C}[t]$. $R$ is a UFD, so it is integrally closed in its field of fractions, which is $K$. $t$ is a root of the monic polynomial $X^2 - 1 - (t^2 - 1)$ in $A[X]$. Thus $R$ is integral over $A$, so that $R$ is the integral closure of $A$ in $K$. □