MATH 7220 Homework 7

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1 Problem 13.1

Let $\phi : A \to B$ be a ring homomorphism, with $\phi^* : Y \to X$ the induced map on $X = \text{Spec}(A), Y = \text{Spec}(B)$. Show that if $\mathfrak{b}$ is an ideal of B, then $\overline{\phi^*(V(\mathfrak{b}))} = V(\mathfrak{b}^c)$.

Proof. Recall that the closure of a set $E \subset X$ is given by $V(I(E))$. With $E = \phi^*(V(\mathfrak{b}))$, we wish to show $I(E) = \mathfrak{b}^c$. We may assume $\mathfrak{b}$ to be a radical ideal, since then $\mathfrak{b}^c$ is as well. $f \in I(E)$ is equivalent to $f(x) = 0$ for all $x \in E$, which is equivalent to $\phi(f)(x) = 0$ for all $x \in V(\mathfrak{b})$, which is equivalent to $\phi(f) \in I(V(\mathfrak{b})) = r(\mathfrak{b}) = \mathfrak{b}$. Thus $f \in I(E)$ is equivalent to $f \in \mathfrak{b}^c$ as desired. □

2 Problem 13.2

Show that $X = \text{Spec}(A)$ is quasi-compact (every open cover has a finite subcover).

Proof. The standard open sets $X_f = X - V(f)$ for $f \in A$ form a basis, so we can take an open covering by X_{f_α}. If the f_α do not generate the ring, then the ideal they generate will be contained in some maximal ideal $\mathfrak{m}$. Then $f_\alpha \in \mathfrak{m}$ for all α, or equivalently, $\mathfrak{m} \notin X_{f_\alpha}$. This contradicts the fact that the X_{f_α} cover X. Thus the f_α generate A. In particular, there is some finite sum $\sum_{i=1}^n g_i f_{\alpha_i} = 1$. Then the f_{α_i} must generate A, which implies that the $X_{f_{\alpha_i}}$ are a finite cover of X. $\square$

3 Problem 13.3

Let $X = \mathrm{Spec}(A)$ where A is Noetherian. Show that every open subset of X is quasi-compact.

Proof. From general topology, we know that a finite union of quasi-compact sets is quasi-compact. Thus we claim that every open subset is a finite union of standard open sets, and that the standard open sets are quasi-compact.

Towards the first claim, take an open set U which is the complement of a closed set $V(\mathfrak{a})$ for an ideal $\mathfrak{a}$. Since A is Noetherian, $\mathfrak{a}$ is finitely generated; $\mathfrak{a} = (f_1, ..., f_n)$. Then $V(\mathfrak{a})$ is the intersection of the $V(f_i)$, which means U is the union of the X_{f_i}.

Now let X_f be a standard open set. X_f can be identified as the spectrum of A_f, the localization of A away from f. Thus by the Problem 13.2, X_f is quasi-compact. $\square$

4 Problem 14.1

Show that the integral closure of $\mathbb{Z}$ in $\mathbb{Q}(\sqrt{5})$ is the set of elements $(a+b\sqrt{5})/2$, where $a, b \in \mathbb{Z}$ have the same parity.

Proof. Let $z = x + y\sqrt{5} \in \mathbb{Q}(\sqrt{5})$. The norm and trace of z are $x^2 - 5y^2$ and $2x$ respectively. For z to be integral, we must have $x^2 - 5y^2, 2x \in \mathbb{Z}$. Thus let $x = a/2$ for $a \in \mathbb{Z}$.

If a is even, then $x \in \mathbb{Z}$. This means $5y^2 \in \mathbb{Z}$, which means $y \in \mathbb{Z}$. Hence $y = b/2$ for some even integer b.

If a is odd, say $a = 2c + 1$, then $x = c + 1/2$. $x^2 = c^2 + c + 1/4$, so we have $1/4 - 5y^2 \in \mathbb{Z}$. Put $y = m/n$ to get $(n^2 - 20m^2)/4n^2 = k\mathbb{Z}$, so that $20m^2 = (1 - 4k)n^2$ or $y^2 = (1 - 4k)/20$. $1 - 4k$ is odd, so the factor of 4 in the denominator cannot cancel out. Thus $y = b/2$ for b an odd integer.

Thus $x + y\sqrt{5} = (a + b\sqrt{5})/2$ for a, b integers of the same parity. □

5 Problem 14.2

Let $f : A \to B$ be an integral morphism of rings. Show that $f^* : \text{Spec}(B) \to \text{Spec}(A)$ is a closed mapping. (See (5.10))

Proof. Recall that f being integral means B is integral over $f(A)$. Let $C = f(A)$. Note that $C \cong A/\ker f$, so the image of $\text{Spec}(C)$ in $\text{Spec}(A)$ is canonically $V(\ker f)$. Thus if the map $\text{Spec}B \to \text{Spec}C$ is closed, we will be done.

Now, C is a subring of B, and B is integral over C. For an ideal $\mathfrak{b}$ in B, we have that $B/\mathfrak{b}$ is integral over $C/\mathfrak{b}^c$. By the prime lifting property, we then have $\text{Spec}(B/\mathfrak{b}) \to \text{Spec}(C/\mathfrak{b}^c)$ is surjective. But again, we can identify $\text{Spec}(B/\mathfrak{b})$ as $V(\mathfrak{b})$ and $\text{Spec}(C/\mathfrak{b}^c)$ as $V(\mathfrak{b}^c)$. The map $\text{Spec}(B/\mathfrak{b}) \to \text{Spec}(C/\mathfrak{b}^c)$ is simply a restriction of the map $\text{Spec}(B) \to \text{Spec}(C)$, so we have shown that the image of $V(\mathfrak{b})$ in $\text{Spec}(C)$ is $V(\mathfrak{b}^c)$, which is closed. Thus $\text{Spec}(B) \to \text{Spec}(C)$ is closed as desired. □

6 Problem 14.3

Let A be a subring of B such that B is integral over A, and let $f : A \to \Omega$ be a morphism of A into an algebraically closed field Ω. Show that f can be extended to a morphism of B into Ω. (Use (5.10)).

Proof. We note that $\ker f$ is prime in A: $A/\ker f$ is isomorphic to the image of f, which is a subring of Ω. Subrings of fields are necessarily integral domains, so $\ker f$ is prime. Then $\mathfrak{p} = \ker f$ can be lifted to a prime $\mathfrak{q}$ in B. Note that $B/\mathfrak{q}$ is an integral domain, and also an integral extension of $A/\mathfrak{p}$. We can factor f through the map $A \to A/\mathfrak{p}$, giving a map $A/\mathfrak{p} \to \Omega$. From proposition 5.23 in Atiyah-Mac Donald, this can be extended to a map $B/\mathfrak{q} \to \Omega$. Pre-composing with the quotient map $B \to B/\mathfrak{q}$ gives the desired extension of f. $\square$